тетрадь/48 листов

1.оп.
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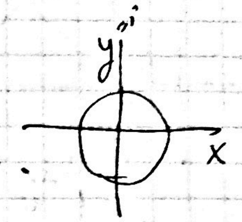
о
по

Семинар

$\left(\frac{1-i}{1+i}\right)^3 = \frac{(1-i)^6}{2^3}$, $\quad 1-i=\sqrt{2}e^{i(-\frac{\pi}{4})}$; $\quad 1+i=\sqrt{2}e^{i\cdot\frac{\pi}{4}} \Rightarrow \frac{1-i}{1+i} = e^{-\frac{3\pi}{2}i} = i$

$\Rightarrow \operatorname{Re} z = 0,$
$\operatorname{Im} z = 1$



$(1+i)^8(1-i\sqrt{3})^{-6}$, $= 16$

№1.4(3)

$z = (-1)^{5}\left(\frac{4}{-1+i\sqrt{3}}\right)^{12} = 2^{12}\cdot e^{-4\pi i}$, главное зн.: $-\pi$ до π, $|z| = 2^{12}$ $\arg z = 0$

№1.4(8)

$z = (1+\cos\alpha) + i\sin\alpha$, $\alpha \in [0, 2\pi]$, $\quad |z| = \sqrt{2+2\cos\alpha} = \sqrt{2}\sqrt{1+\cos\alpha} = 2\left|\cos\frac{\alpha}{2}\right|$
$\quad x \qquad\qquad y$

$x \geq 0$, $\quad x = 0 \Rightarrow \alpha = \pi \Rightarrow y = 0 \Rightarrow$ невозможен
$x > 0 \Rightarrow \arg z = \operatorname{arctg}\frac{y}{x} = \operatorname{arctg}\left(\frac{\sin\alpha}{1+\cos\alpha}\right) = \operatorname{arctg} \operatorname{tg}\frac{\alpha}{2}$

№1.5(3)

$z^2 + z|z| + |z|^2 = 0$, $\quad (z+|z|)^2 = z\cdot|z|$, $\quad z = |z|\cdot e^{i\varphi} \Rightarrow |z|^2(e^{2i\varphi} + e^{i\varphi} + 1) = 0$

$\Rightarrow |z|^2(\cos 2\varphi + i\sin 2\varphi + \cos\varphi + i\sin\varphi + 1) = 0 \Rightarrow$ 1) $z = 0$

2) $\begin{cases} \cos 2\varphi + \cos\varphi + 1 = 0 \\ \sin 2\varphi + \sin\varphi = 0 \end{cases}$

$\begin{cases} 2\cos^2\varphi + \cos\varphi = 0 \\ 2\sin\varphi\cos\varphi + \sin\varphi = 0 \end{cases}$

2.1) $\cos\varphi = 0$, $\varphi = \frac{\pi}{2}$, $\varphi = -\frac{\pi}{2}$,

$\left.\begin{array}{l} z = r\cdot e^{\frac{\pi}{2}i} = ir \\ z = r\cdot e^{-\frac{\pi}{2}i} = -ir \end{array}\right| \Rightarrow$ нет реш.

2.2) $2\cos\varphi + 1 = 0 \Rightarrow \varphi = \pm\frac{2\pi}{3} + 2\pi k, k\in\mathbb{Z}$

$\Rightarrow$ Ответ: $z = re^{\pm\frac{2\pi}{3}i}$

№1.8(1,2)

1) $|z_1+z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1|\cdot|z_2|\cos(\arg z_1 - \arg z_2)$

$z_1 = x_1 + iy_1$, $\bar{z}_1 = x_1 - iy_1$, $\Rightarrow z\bar{z} = |z|^2 \Rightarrow |z_1+z_2|^2 = (z_1+z_2)\overline{(z_1+z_2)} = z_1\bar{z}_1 + z_1\bar{z}_2 +$
$+ z_2\bar{z}_2 + \bar{z}_1 z_2 =$
$= |z_1|^2 + |z_2|^2 +$
$+ |z_1|\cdot e^{i\arg z_1}\cdot|z_2|\cdot$
$\cdot e^{-i\arg z_2}$ $\oplus$

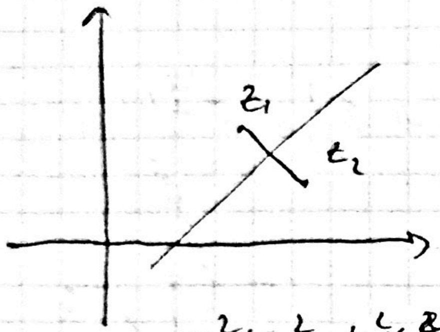
$\oplus$ $|z_1|\cdot e^{-i\arg z_1}\cdot|z_2|\cdot e^{i\arg z_2} = \ldots$ ч.т.д.

2) ~~$|z_1-z_2| \geq ||z_1|-|z_2||$~~ $|z_1+z_2|^2 + |z_1-z_2|^2 \ominus 2(|z_1|^2 + |z_2|^2)$

$\ominus$ $2|z_1|^2 + 2|z_2|^2$

№1.14

$\left|\frac{z-z_1}{z-z_2}\right| = k$, $k > 0$, $z \in \mathbb{C}$



$z = x + iy$
$z_1 = x_1 + iy_1$
$z_2 = x_2 + iy_2$

$$\left|\frac{z-z_1}{z-z_2}\right|^2 = k^2 \Rightarrow (x-x_1)^2 + (y-y_1)^2 = k^2\left((x-x_2)^2 + (y-y_2)^2\right)$$

$$x^2 - 2xx_1 + x_1^2 + y^2 - 2yy_1 + y_1^2 = k^2(x^2 - 2xx_2 + x_2^2 + y^2 - 2yy_2 + y_2^2)$$

$$x^2(1-k^2) - 2x(x_1 - k^2x_2) + y^2(1-k^2) - 2y(y_1 - k^2y_2) + x_1^2 + y_1^2 - k^2x_2^2 - k^2y_2^2 = 0,$$

$$(1-k^2)\left(x - \frac{x_1-k^2x_2}{1-k^2}\right)^2 - \left(\frac{x_1-k^2x_2}{1-k^2}\right)^2 + (1-k^2)\left(y - \frac{y_1-k^2y_2}{1-k^2}\right)^2 - \frac{(y_1-k^2y_2)^2}{1-k^2} + x_1^2 + y_1^2 - k^2x_2^2 - k^2y_2^2 = 0.$$

$$\left(x - \frac{x_1-k^2x_2}{1-k^2}\right)^2 + \left(y - \frac{y_1-k^2y_2}{1-k^2}\right)^2 = \frac{(x_1-k^2x_2)^2}{(1-k^2)^2} + \frac{(y_1-k^2y_2)^2}{(1-k^2)^2} + \frac{k^2x_2^2 + k^2y_2^2 - x_1^2 - y_1^2}{1-k^2} =$$

$$= \frac{x_1^2 - 2k^2x_2x_1 + k^4x_2^2 + y_1^2 - 2k^2y_1y_2 + k^4y_2^2 + k^2x_2^2 + k^2y_2^2 - x_1^2 - y_1^2 - k^4x_2^2 - k^4y_2^2 + k^2x_1^2 + k^2y_1^2}{(1-k^2)^2} =$$

$$= \frac{k^2}{(1-k^2)^2}\left((x_1-x_2)^2 + (y_1-y_2)^2\right) = \frac{k^2}{(1-k^2)^2}|z_1 - z_2|^2, \quad k \neq 1 \Rightarrow$$

Ур-е окр.: $|z - z_0| = R$; $R = \frac{k}{|1-k^2|}|z_1 - z_2|$; $z_0 = \frac{z_1 - k^2z_2}{1-k^2}$ — Окружность Аполлония $k \neq 1, 0$

№ 1.11 (1,2)

1) $\cos x + r\cos(x+\alpha) + \_ + r^n\cos(x+n\alpha) = S_1$, $S_1 + iS_2$

2) $\sin x + r\sin(x+\alpha) + \_ + r^n\sin(x+n\alpha) = S_2$

e^{ix} $re^{\alpha i}$; $z_1 = \cos x + i\sin x$, $z_2 = r(\cos\alpha + i\sin\alpha)$

$$S_1 + iS_2 = z_1 + z_1z_2 + \_ + z_1z_2^n = z_1(1 + z_2 + \_ + z_2^n) = z_1\left(\frac{1-z_2^{n+1}}{1-z_2}\right)$$

Д/з

№ 1.2 (8,9)

8) $\left(\frac{1-i\,tg\alpha}{1+i\,tg\alpha}\right)^n = \left(e^{-2i\,tg\alpha}\right)^n = e^{-2n\,tg\alpha\, i} \Rightarrow$

9) $(1 + \cos\alpha + i\sin\alpha)^8$

№ 1.4 (3,10)

3) $z = \frac{1+\cos\alpha + i\sin\alpha}{1+\cos\alpha - i\sin\alpha}$, $0 \leq \alpha \leq 2\pi$, $\alpha \neq \pi$ $\qquad |z| \geq 1$, $\sin^2\alpha = \frac{1 - \cos 2\alpha}{2}$

$$\arg z = \operatorname{arctg}\left(\frac{\sin\alpha}{1+\cos\alpha} + \frac{\sin\alpha}{1+\cos\alpha}\right) = \operatorname{arctg}\left(\frac{2\sin\alpha}{1+\cos\alpha}\right) = \operatorname{arctg}\left(\frac{2\cdot 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{2\cos^2\frac{\alpha}{2}}\right) = \operatorname{arctg}\left(2\,tg\frac{\alpha}{2}\right)$$

11) $\underbrace{z^2+z}_{z'}$, $|z|=1 \Rightarrow z=x+iy$, $\sqrt{x^2+y^2}=1$, $x^2+y^2=1$
$z=|r|(\cos\varphi+i\sin\varphi)$

$z' = \cos 2\varphi + i\sin 2\varphi + \cos\varphi + i\sin\varphi = \underbrace{\cos 2\varphi+\cos\varphi}_{2\cos\frac{3\varphi}{2}\cos\frac{\varphi}{2}} + i(\underbrace{\sin 2\varphi+\sin\varphi}_{2\sin\frac{3\varphi}{2}\cos\frac{\varphi}{2}})$ $\quad |z'| = \sqrt{2\sin\varphi\sin 2\varphi + 2\cos\varphi\ldots} = 2\cos\varphi/2$.

$x \quad , \quad y$

$x>0$ $\operatorname{arctg}\frac{y}{x}$ $\quad \frac{y}{x} = \operatorname{tg}\frac{\varphi}{2}\cdot\operatorname{tg}\frac{3\varphi}{2}$
$x<0$, $y<0$
$y>0$

№ 1.5(1,4,5)

1) $z^2=i$, $r^2(\cos 2\alpha + i\sin 2\alpha) = 1(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2})$

4) $|z| = z+2i+1$, $\quad z=x+iy \Rightarrow \sqrt{x^2+y^2} = x+iy+2i+1$,
$z=x-2i \Rightarrow \sqrt{x^2+4} = x+1$, $\cancel{x^2}+4=\cancel{x^2}+2x+1$, $x=\frac{3}{2}$.

5) $\bar z = z^{n-1}, n\in\mathbb{N}$, $\quad re^{-i\varphi} = r^{n-2}e^{i(n-1)\varphi} \Rightarrow \left[\begin{array}{l} r=0 \\ r=1; \quad -\varphi=(n-1)\varphi+2\pi k, k\in\mathbb{Z} \end{array}\right.$
$0=\varphi n+2\pi k, k\in\mathbb{Z}$
$\varphi = +\frac{2\pi k}{n}, k\in\mathbb{Z}$

№ 1.40(1,2,3)

1) $\left|\sum_{i=1}^{n} z_i\right| \le \sqrt{n\sum_{i=1}^{n}|z_i|^2}$, $\quad n=2$: $|z_1+z_2| \overset{?}{\le} \sqrt{2(|z_1|+|z_2|)}$
$z_i=$ $\quad \sqrt{(x_1+x_2)^2+(y_1+y_2)^2} \overset{?}{\le} \sqrt{2(\sqrt{x_1^2+y_1^2}+\sqrt{x_2^2+y_2^2})}$

~~Для $a_i\in\mathbb{R}$~~. Из нер-ва Ляпунова: $\forall\ 0<\alpha\le\beta$: $(M|\xi|^\alpha)^{1/\alpha} \le (M|\xi|^\beta)^{1/\beta}$

$\Rightarrow \sqrt{n\sum|z_i|^2} = n\sqrt{\frac{\sum|z_i|^2}{n}} \ge n\cdot\frac{\sum|z_i|}{n} \ge |\sum z_i|$ ч.т.д.

2) $\sum_{k=1}^{n}|z_k| \le n^{\frac{p-1}{p}}\left(\sum_{k=1}^{n}|z_k|^p\right)^{1/p}$; $p\ge 1$ Нер-во Гёльдера $\quad n=\sum_{k=1}^{n} 1^{\frac{p}{p-1}}$.

3) Также как 1).

Инд. работа

2') $\sum_{k=1}^{n}|z_k|=1$; (т.к. отн-но одн. сумм.)

$1 \le \left(\sum_{k=1}^{n}|z_k|^p\right) \Rightarrow |z_k|\le 1 \quad |z_k|^p\ge 1$, $p\in(0;1)$ $\Bigg| \Rightarrow \left|\sum_{k=1}^{n}|z_k|^p\right| \ge \sum_{k=1}^{n}|z_k| \ge 1$

Функ-и компл. перемен.

$z=x+iy$, $e^z=e^x(\cos y+i\sin y)$; $|e^z| = e^x = e^{\operatorname{Re} z}$; $e^{z_1}=e^{z_2}$; $\quad z_1=x_1+iy_1$,
$z_2=x_2+iy_2$

$e^{x_1}(\cos y_1 + i\sin y_1) = e^{x_2}(\cos y_2 + i\sin y_2) \Rightarrow \begin{cases} e^{x_1}\cos y_1 = e^{x_2}\cos y_2 \\ e^{x_1}\sin y_1 = e^{x_2}\sin y_2 \end{cases}$

$\Rightarrow \begin{cases} e^{2x_1} = e^{2x_2} \Rightarrow x_1 = x_2 \\ \cos y_1 = \cos y_2 \quad y_1 = y_2 + 2\pi k,\ k \in \mathbb{Z} \\ \sin y_1 = \sin y_2 \end{cases}$

$\Rightarrow z_1 = z_2 + i2\pi k,\ k \in \mathbb{Z}$

$\nexists \lim\limits_{z\to\infty} e^z = \lim\limits_{\substack{x\to\infty \\ y\to\infty}} e^x(\cos y + i\sin y)$; $\lim\limits_{z\to\infty} f(z) = \omega$ $\forall \varepsilon > 0\ \exists \delta > 0\ \forall z: |z| > \delta \Rightarrow |f(z) - \omega| < \varepsilon$

(уже по действ.)

e^z - ф. не огр.; $e^z = 0$ не имеет реш.

$\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$; $(\cos z)' = \frac{1}{2}(ie^{iz} - ie^{-iz}) = -\sin z$

$\cos z_1 = \cos z_2 \Rightarrow e^{iz_1} + e^{-iz_1} = e^{iz_2} + e^{-iz_2}$; $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$,

$e^{ix_1 - y_1} + e^{y_1 - ix_1} = e^{ix_2 - y_2} + e^{y_2 - ix_2}$

$\Rightarrow e^{-y_1}(\cos x_1 + i\sin x_1) + e^{y_1}(\cos x_1 - i\sin x_1)$

$e^{iz_1} + e^{-iz_1} = e^{iz_2} + e^{-iz_2}$

$e^{iz_1} + \dfrac{1}{e^{iz_1}} = e^{iz_2} + \dfrac{1}{e^{iz_2}} \Rightarrow \dfrac{e^{iz_1} - e^{iz_2}}{e^{iz_1}e^{iz_2}} = e^{iz_1} - e^{iz_2}$

$\Rightarrow (e^{iz_1} - e^{iz_2})\left(1 - \dfrac{1}{e^{i(z_1+z_2)}}\right) = 0$

$\Rightarrow \left[\begin{array}{l} e^{iz_1} = e^{iz_2} \Rightarrow z_1 = z_2 + 2\pi k i,\ k \in \mathbb{Z} \\ e^{i(z_1+z_2)} = e^{2\pi m i},\ m \in \mathbb{Z} \end{array}\right.$

$z_1 + z_2 = 2\pi m,\ m \in \mathbb{Z}$

$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

$\cos z_1 + \cos z_2 \overset{?}{=} 2\cos\left(\frac{z_1+z_2}{2}\right)\cos\left(\frac{z_1-z_2}{2}\right)$

$\alpha = \frac{z_1+z_2}{2}$ $z_1 = \alpha + \beta$

$\beta = \frac{z_1-z_2}{2}$ $z_2 = \alpha - \beta$ $\Rightarrow$ через exp все выходит

$\cos z = 0$, $e^{iz} + e^{-iz} = 0$, $e^{iz} = e^{-iz} \cdot e^{(\pi + 2\pi k)i}$

$\Rightarrow z = -z + \pi + 2\pi k$ (учёт(к)

$\Rightarrow z = \frac{\pi}{2} + \pi k,\ k \in \mathbb{Z}$.

$|\cos z| = \left|\dfrac{e^{iz} + e^{-iz}}{2}\right| = \dfrac{1}{2}\left|\dfrac{e^{ix-y} + e^{-ix+y}}{1}\right| =$

$= \frac{1}{2}\left|e^{-y}(\cos\_ + i\sin\_) + e^{y}(\cos\_ - i\sin\_)\right| =$

$= \frac{1}{2}\sqrt{(e^{-y}\cos x + e^{y}\cos x)^2 + (e^{-y}\sin x - e^{y}\sin x)^2} =$

$= \frac{1}{2}\sqrt{\cos^2 x(e^{2y} + e^{-2y} + 2) + \sin^2 x(e^{2y} + e^{-2y} - 2)}$

$= \frac{1}{2}\sqrt{e^{2y} + e^{-2y} + 2\cos 2x} = \frac{\sqrt{2}}{2}\sqrt{\frac{1}{2}\operatorname{ch} 2y + \frac{1}{2}\cos 2x}$

$|\cos z|$ — неогр.

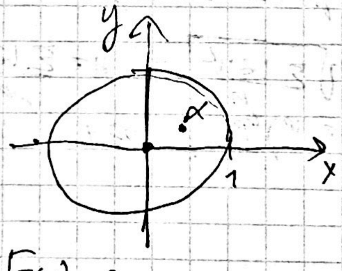
3.45

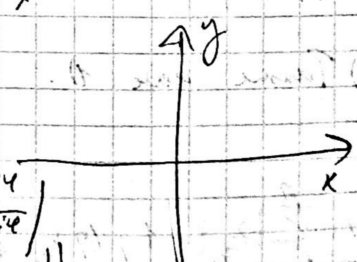
$|\alpha| < 1$; $f(z) = \dfrac{\alpha - z}{1 - \bar{\alpha} z}$ $\{|z| \le 1\}$

$z = x_1 + iy_1$

$\Downarrow$

$f(z) = \dfrac{\alpha - x_1 - iy_1}{\ldots}$





$F(\alpha) = 0$

$z = re^{i\varphi}$

$|r| \le 1$ $|F(z)| = \left|\dfrac{\alpha - re^{i\varphi}}{1 - \bar{\alpha} re^{i\varphi}}\right|$

внутр. → внутр.

гран. → гран.

$\|$

$|e^{i\varphi}| \left|\dfrac{\alpha e^{-i\varphi} - r}{1 - \bar{\alpha} re^{i\varphi}}\right| \ominus$

$\ominus \left|\dfrac{\alpha e^{-i\varphi} - r}{1 - \bar{\alpha} re^{i\varphi}}\right| \overset{?}{\le} 1 = \left|\dfrac{(\alpha e^{-i\varphi} - r)(1 - \alpha r e^{-i\varphi})}{(1 - \bar{\alpha} r e^{i\varphi})(1 - \alpha r e^{-i\varphi})}\right|$

$\ominus$

$z=\rho e^{i\theta}$

$\stackrel{=}{} \left| \dfrac{(A)}{1+|\alpha|^2 r^2 - r[\alpha e^{-i\varphi} + \bar\alpha e^{i\varphi}]} \right| \stackrel{=}{}$

$z=\rho e^{-i\theta} \Rightarrow [\alpha e^{-i\varphi} + \bar\alpha e^{i\varphi}] = \rho e^{i(\theta - \varphi)} + \rho e^{i(\varphi - \theta)} =$

$= 2\rho\cos(\theta-\varphi)$

$\stackrel{=}{} (A) = \alpha e^{-i\varphi} - \alpha^2 \bar r e^{-2i\varphi} - r + \alpha r \bar e^{-i\varphi}$

$f(z) = \dfrac{\alpha - z}{1-\bar\alpha z} = \dfrac{(\alpha-z)(1-\alpha\bar z)}{(1-\bar\alpha z)(1-\alpha\bar z)} \stackrel{=}{}$

$\stackrel{=}{} \dfrac{\alpha - \alpha^2\bar z - z + \alpha|z|^2}{1-\alpha\bar z - \bar\alpha z + |\alpha|^2|z|^2} = \dfrac{\alpha(1+|z|^2) - \alpha^2\bar z - \bar z}{1+|\alpha|^2|z|^2 - \alpha\bar z - \bar\alpha z}$

$|f(z)| = \left|\dfrac{\alpha - z}{1-\bar\alpha z}\right| \stackrel{?}{<} 1$ или $\left|\dfrac{\alpha-z}{1-\bar\alpha z}\right|^2 < 1$

$|z|<1$

или $|\alpha - z|^2 \vee |1-\bar\alpha z|^2$

$(\alpha - z)(\bar\alpha - \bar z) \vee (1-\bar\alpha z)(1-\alpha\bar z)$

$\Rightarrow |\alpha|^2 + |z|^2 \underset{<}{\overset{?}{\vee}} 1 + |\alpha|^2|z|^2$

$|\alpha|^2(1-|z|^2) \vee 1 - |z|^2 \; ; \; (1-|z|^2)(1-|\alpha|^2) \underset{=}{>} 0$

Внешние!

$\dfrac{\alpha - z}{1-\bar\alpha z} = w \Rightarrow$ ~~$\alpha - z = w - w\bar\alpha z$~~

$z = \dfrac{w-\alpha}{-1+w\bar\alpha} = \dfrac{\alpha - w}{1-w\bar\alpha}$

Д/З

№3.44

$f(z) = \prod\limits_{k=1}^{n} \dfrac{\alpha_k - z}{1-\bar\alpha_k z}$ $|\alpha_k|<1$, непр в $|z|<1$

и $|f(z)| \le 1$ при $|z| \le 1$

Д/З 3.45

№3.38

$\lim\limits_{z\to\infty} e^{z^2} = \infty$, если $z \in \{z \in \mathbb{C} : |\arg z| \le \alpha < \frac{\pi}{4}\}$ $|\arg z - \pi| \le \alpha > \pi/4$!

$|e^{z^2}| = |e^{(\ldots)}| \Rightarrow z^2 = r^2 e^{i2\varphi}$

$z = r e^{i\varphi}$

$\Rightarrow e^{r^2 e^{i2\varphi}} =$

$= e^{r^2(\cos 2\varphi + i\sin 2\varphi)} \Rightarrow$

$\Rightarrow |e^{r^2(\cos 2\varphi + i \sin 2\varphi)}| - ;$

$\lim\limits_{z\to\infty} e^{z^2} = \infty \Leftrightarrow \forall \varepsilon > 0 \; \exists \delta > 0 : \forall z \in \mathbb{C} :$
$|z| > \delta \;\; |e^{z^2}| > \varepsilon.$

$|e^{z^2}| = e^{\operatorname{Re} z^2} = e^{x^2 - y^2}$

$z = x+iy \Rightarrow z^2 = (x+iy)(x+iy) \Rightarrow \operatorname{Re} z^2 = x^2 - y^2$

a) $e^{x^2-y^2} > \underset{e^{\ln\varepsilon}}{\varepsilon}$, $x^2 - y^2 > \ln\varepsilon$

$(\underbrace{x^2+y^2}_{\delta^2} - \underbrace{2y^2}_{2\delta^2}) > \ln\varepsilon$

$\Rightarrow \delta = \sqrt{\dfrac{\ln\varepsilon}{2}}$ и т.д.

№3.34

$f(z) = e^{-1/z}$ $\{0<|z|<R,\ |\arg z| < \alpha\}$

1) $\alpha = ?$ $e^{-1/z}$ непр в этой обл. ?
непр. как Im и Re fun.

$z \ne 0 \Rightarrow \forall \alpha \in \mathbb{R}$

2) $\alpha < \pi/2$.

$\left| e^{-1/z_1} - e^{-1/z_2} \right| =$
$= |e^{-1/z_2}| \, |e^{\frac{1}{z_2} - \frac{1}{z_1}} - 1| <$
$< |e^{-\frac{1}{z_2}}| \, |e^{\frac{1}{z_2} - \frac{1}{z_1}}$

№3.37

$f(z)$ р.н. на X : $\forall \varepsilon > 0 \; \exists \delta > 0 : |x_1 - x_2| < \delta$
$\Rightarrow |f(x_1) - f(x_2)| < \varepsilon$

№1) $\dfrac{1}{1-z}$ Пусть $z = \dfrac{1}{n} \Rightarrow \dfrac{1}{1-\frac{1}{n}} = \dfrac{n}{n-1} =$

$= 1 + \dfrac{1}{n-1} \ge \varepsilon \Rightarrow$ не р.н. на $|z|<1$.

2) $\dfrac{1}{1+z^2}$ $\exists\, z = \dfrac{1}{\sqrt{n}} \Rightarrow$ аналог.

3) $e^{1/(z-1)}$ нигде $\Rightarrow$ р.н.

~~4) $e^{-1/z-1}$~~ 5), 6 7), 9)

№3.36 $\int$ 3) р.н., 5) р.н., 6), 7), 8)

Кл. работа

№ 5.1(4).

1) $f(z)=|z|^2 \Rightarrow \begin{cases} u'_x = v'_y \\ u'_y = -v'_x \end{cases} \Rightarrow$ diff. $z=0$

(2) $f(z)=z^2\bar{z} = (x+iy)^2(x-iy) =$

$z=x+iy \mid u(x,y)=x(x^2+y^2)$

$v(x,y)=y(x^2+y^2)$

$\frac{\partial u}{\partial x} = 3x^2+y^2 \quad \frac{\partial u}{\partial y} = 2xy$

$\frac{\partial v}{\partial x} = 2xy \quad \frac{\partial v}{\partial y} = x^2+3y^2$

т. $(0,0)$

№ 5.4

$f(z)\in A(D)$, $D\subset\mathbb{C}$ ~~и~~ ~~всюду~~ $\mathrm{Re}\, f(z)=\mathrm{const}$

$\Rightarrow f(z)$ всюду в D const.

из res. усл.

№ 5.8(а)

$f(z)=u+iv$, u,v — дифф. в $D\subset\mathbb{C}$

~~и~~ $\exists \lim\limits_{\Delta z\to 0} \mathrm{Re}\frac{\Delta f}{\Delta z} = a$ в D $\Rightarrow f(z)$ анал.

$\Delta f = \Delta u + i\Delta v$; $\quad \mathrm{Re}\frac{\Delta f}{\Delta z} = \mathrm{Re}\frac{\Delta u + i\Delta v}{\Delta x + i\Delta y} =$

$\Delta z = \Delta x + i\Delta y$;

$= \mathrm{Re}\frac{(\Delta u + i\Delta v)(\Delta x - i\Delta y)}{\Delta x^2+\Delta y^2} = \frac{1}{\Delta x^2+\Delta y^2}(\Delta u\cdot\Delta x + \Delta v\,\Delta y)$

$\lim\limits_{\substack{\Delta z\to 0\\ \Delta x\to 0\\ \Delta y\to 0}} \ldots = a$

$\Delta u = u'_x\cdot\Delta x + u'_y\cdot\Delta y + \bar{o}_1(1)\cdot\Delta x + \bar{o}_2(1)\cdot\Delta y$

$\Delta v = v'_x\cdot\Delta x + v'_y\cdot\Delta y + \bar{o}_3(1)\cdot\Delta x + \bar{o}_4(1)\cdot\Delta y$.

$a = \lim\limits_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{u'_x\cdot\Delta x^2 + u'_y\cdot\Delta x\cdot\Delta y + \bar{o}(1)\Delta x^2 + \bar{o}_2(1)\Delta x\Delta y}{\Delta x^2+\Delta y^2}$

$= \big\rbrace\, \Delta y = \alpha\Delta x \Rightarrow$

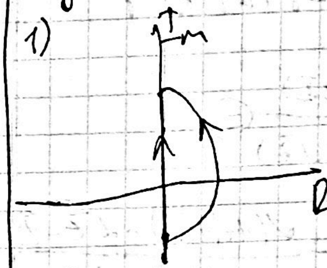
$a = \lim\limits_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \ldots = a(1+\alpha^2) = u'_x + \alpha u'_y + \alpha v'_x + \alpha^2 v'_y$ ...

$= \begin{cases} a = v'_y \\ 0 = u'_y + v'_x \\ a = u'_x \end{cases} \quad \begin{cases} u'_x = v'_y \\ u'_y = -v'_x \end{cases}$ ч.т.д.

Интегр. функ-й.

№ ~~6.8~~ 6.5

$\int_\gamma |z|\,dz$

1) 

$z=it$, $-1\le t\le 1$

$dz=i\,dt$

$\int_\gamma |z|\,dz = \int_{-1}^{1}\sqrt{x^2+y^2}\, i\,dt =$

$= i\int_{-1}^{1}|t|\,dt = 2i\int_0^1 t\,dt = 2i\frac{1}{2}t^2\Big|_0^1 = i$

2) $z=e^{i\varphi}$, $\int_\gamma |z|\,dz = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} ie^{it}\,dt = e^{it}\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} =$

$= e^{i\frac{\pi}{2}} - e^{-i\frac{\pi}{2}} = 2i$

Функ. не анал. $\Rightarrow$ разные интегралы.

№ 6.9

$\rho>0$, $|z|=\rho$ $\quad \int\limits_{|z|=\rho} z^n\,dz = \begin{cases} 2\pi i, & n=-1 \\ 0, & n=0,\pm1,\pm2 \end{cases}$

$|z|=\rho>0 \mid \int\limits_{|z|=\rho} z^n\,dz = \int_0^{2\pi}\rho^n e^{in\varphi}\cdot\rho i e^{i\varphi}\,d\varphi =$

$z=\rho e^{i\varphi}$, $\varphi\in[0,2\pi] \Rightarrow dz=\rho i e^{i\varphi}d\varphi$

$= \rho^{n+1} i \int_0^{2\pi} e^{i(n+1)\varphi}\,d\varphi = \begin{cases} ①, & n=-1 \\ ②, & n\ne -1 \end{cases}$

① $2\pi i \rho^{n+1}$

② $i\rho^{n+1}\cdot\frac{e^{(n+1)i\varphi}}{(n+1)i}\cdot(-1)\Big|_0^{2\pi} = 0$.

№ 6.21

$f(z)$ — непр. в окр. z_0

$D\to z_0$, $z\ne 0$ $\quad \lim\limits_{\rho\to 0}\int\limits_{|z-z_0|=\rho}\frac{f(z)dz}{z-z_0} = 2\pi i f(z_0) =$

$z=\rho e^{i\varphi}+z_0$, $\varphi\in[0,2\pi]$

$dz = i\rho e^{i\varphi}d\varphi$

$\int_0^{2\pi} \frac{f(z_0+\rho e^{i\varphi}) i\rho e^{i\varphi} d\varphi}{\rho e^{i\varphi}}; \int_0^{2\pi} f(z_0+\rho e^{i\varphi}) d\varphi$

$\left|\int_0^{2\pi} f(z_0+\rho e^{i\varphi}) d\varphi - 2\pi f(z_0)\right| = \left|\int_0^{2\pi} (f(z_0+\rho e^{i\varphi}) - f(z_0)) d\varphi\right|$

$\leq \int_0^{2\pi} |f(z_0+\_) - f(z_0)| d\varphi \xrightarrow[\rho\to 0]{\text{непр.}} 0$

$\int$ 6.27

D - выпукл. и $\operatorname{Re} f(z) \geq M > 0, z \in D$

$\Rightarrow \left|\int_{z_1}^{z_2} f(z) dz\right| \geq M|z_2 - z_1|$

$z_1, z_2 \in D$; $f(z) = \cancel{x+iy}\ u + iv$

$\int_{z_1}^{z_2} \cancel{(u+iv)dz} = \quad z = tz_1 + (1-t)z_2,\ t \in [0,1]$

$\int_0^1 f(tz_1 + (1-t)z_2)\, \underbrace{d(tz_1 + (1-t)z_2)}_{(z_1-z_2)dt} =$

$= (z_1 - z_2) \int_0^1 f(tz_1 + (1-t)z_2) dt$

$\left|\int_0^1 f(tz_1 + (1-t)z_2) dt\right| = \left|\int_0^1 \operatorname{Re} f(\_) dt + i\int_0^1 \operatorname{Im} f() dt\right|$

$= \sqrt{\left(\int_0^1 \operatorname{Re} f() dt\right)^2 + \left(\int_0^1 \operatorname{Im} f() dt\right)^2} \geq \left|\int_0^1 \operatorname{Re} f() dt\right|$ $\geq M$

$\geq M$.

Уточнить. Т. Коши непр. $\xrightarrow{+\infty} a$ | $\Rightarrow$ р.н. $(b,+\infty)$

Д/р

огр. монот. непр.
$\Rightarrow$ р.н.
f(x) - дифф. произв. $\Rightarrow$ р.н.

~~6.28~~ $\int$ 5.1 (6, 7, 9)

6) x^2y^2 $\quad u'_x = +v'_y \quad u'_y = -v'_x$ $\Rightarrow 2xy^2 = 0$, $2yx^2 = 0$

$x = 0, y \in \mathbb{R}$

$y = 0, x \in \mathbb{R}$

две прямые

7) $x^2 + iy^2 \Rightarrow 2x = 2y$, $0 = 0$ $\quad x = y$

$\Rightarrow$

9) $2xy - i(x^2 - y^2) \Rightarrow 2y = 2y$, $2x = 2x$; $y \neq 0$, $x \neq 0$

для полинома

$\int$ 6.6

$\int_{|z|=1} |z-1| dz \qquad z = e^{i\varphi},\ 0 \leq \varphi \leq 2\pi$

$= \int_0^{2\pi} dz = ie^{i\varphi} d\varphi$

$\int |e^{i\varphi} - 1| \cdot ie^{i\varphi} d\varphi$

$|e^{i\varphi} - 1| = |\cos\varphi + i\sin\varphi - 1| = \sqrt{(\cos\varphi - 1)^2 + \sin^2\varphi}$

$= \sqrt{1 + 1 - 2\cos\varphi} = \sqrt{2}\sqrt{1 - \cos\varphi}$

$I = i\sqrt{2} \int_0^{2\pi} \sqrt{1 - \cos\varphi}\, e^{i\varphi} d\varphi =$

$= i2\left(\int_0^{\pi} \sin\varphi\, e^{i\varphi} d\varphi + \int_{\pi}^{2\pi} \sin\varphi\, e^{i\varphi} d\varphi\right)$

$I^* = \int \sin\varphi\, e^{i\varphi} d\varphi = \frac{1}{i}\int \sin\varphi\, d(e^{i\varphi}) = \frac{1}{i}\left(\sin\varphi\, e^{i\varphi}\right|$

$- \int e^{i\varphi} \cos\varphi d\varphi)$

$\frac{1}{i}\int \cos\varphi\, de^{i\varphi} = \frac{1}{i}\left(\cos\varphi\, e^{i\varphi}\right| + \int e^{i\varphi} \sin\varphi d\varphi)$

$\Rightarrow I^* = \frac{1}{i}(\sin\varphi\, e^{i\varphi}| - \frac{1}{i}(\cos e^{i\varphi}| + I^*))$

$\Rightarrow -I^* = i\sin\varphi\, e^{i\varphi}| - \cos e^{i\varphi}| - I^*$

$\int \sin\varphi\, e^{i\varphi} d\varphi = -(\cancel{\int e^{i\varphi}\cos\varphi}) \quad \frac{e^x \sin x - e^x \cos x}{2}$

$\Rightarrow I =$

$\int$ 6.10

$\int_{|z|=\rho} \frac{|dz|}{|z^2 - a^2|} \leq \frac{2\pi\rho}{|\rho^2 - |a|^2|}$, $0 < \rho \neq |a|$.

$\int_\gamma \frac{1}{|z^2 - a^2|} dz$

$z = \rho \cdot e^{i\varphi}$, $dz = \rho i e^{i\varphi} d\varphi$

$\int$ 6.16

$f(z)$ - непр. на $\operatorname{Im} z \geq 0$

и $|f(z)| \leq M|z|^m$, $\operatorname{Im} z \geq 0$

$\Rightarrow \left|\int_\gamma f(z) e^{iz} dz\right| \leq \pi M R^m$, γ - полуокр. $|z| = R$, $\operatorname{Im} z \geq 0$

$\leq \int_\gamma |f(z)| \cdot |e^{iz}|\, |dz| \leq M \int_\gamma |z|^m |dz|$

$|e^{iz}| = 1 \Leftrightarrow \mathrm{Im}\, z = \dots$

№ 6.22

$f(z)$ непр. в $0<|z-z_0|<R$

$M(r)=\max\limits_{|z-z_0|=r<R}|f(z)|$, $\lim\limits_{r\to 0} rM(r)=0$

$\Downarrow$

$\lim\limits_{r\to 0}\oint\limits_{|z-z_0|=r} f(z)dz = 0.$

$(*) \le \oint\limits_{|z-z_0|=r}|f(z)||dz| \le M(r)\cdot 2\pi r \to 0$

№ 6.28

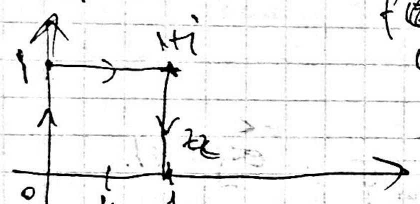
$f(z)$ анал. в огр. выпукл. — анал. кр. раб.

Контрпример: усл. анал. заменить непрерывностью.

$f(z)\in C(D)$, D — выпукла

$\mathrm{Im} f(z) \ge M>0$, $\forall z_1, \forall z_2\in D$

$\left|\int_{z_1}^{z_2} f(z)dz\right| \overset{!}{\ge} M|z_2-z_1|$



$f(z)=u(x,y)+iv(x,y)$

$u(x,y)=1$

$v(x,y)=\begin{cases}0, & x>1/2\\ 1-2x, & x\le 1/2\end{cases}$

$\int_\gamma f(z)dz = \int_\gamma u(x,y)dx - v(x,y)dy + i\int_\gamma u(x,y)dy + v(x,y)dx,$

$\gamma_1: \int_{\gamma_1} f(z)dz = \int_0^1 -1dy + i\int_0^1 1dy = -1+i$

$\gamma_2: \int_{\gamma_2} f(z)dz = \int_0^1 1dx + i\int_0^{1/2}(1-2x)dx =$

$= 1+i(x-x^2)\Big|_0^{1/2} = 1+\frac{1}{4}i$

$\gamma_3: \int f(z)dz = -i$, $M|z_2-z_1| = 1\cdot 1$

кр. работа

№ 7.5

$I=\oint \frac{dz}{z^2+9}$ γ — замкн. крив.

1) $3i \in \mathrm{int}\,\gamma$, $-3i \notin \overline{D}_\gamma$

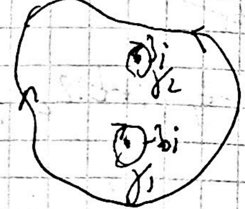
$I = \frac{2\pi i}{2\pi i}\int_\gamma \frac{dz}{(z-3i)(z+3i)}$, $\frac{1}{z+3i}=f(z)$

$\frac{1}{2\pi i}\int_\gamma \frac{f(\xi)d\xi}{\xi - z} = f(z) = 2\pi i\cdot\frac{1}{6i} = \frac{\pi}{3}$

2) $-3i\in \overline{D}_\gamma$, $3i \notin \overline{D}_\gamma$

$\int_\gamma \frac{dz}{(z-3i)(z+3i)} = -\frac{1}{6i}2\pi i = -\frac{\pi}{3}$

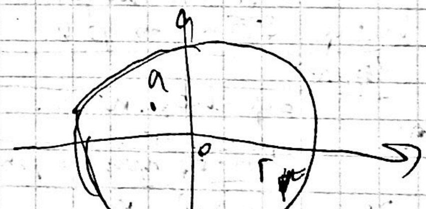
3) $-3i, 3i \in D_\gamma$



$I = \int_\gamma \frac{dz}{z^2+9} = \int_{\gamma_1}\frac{dz}{\dots} + \int_{\gamma_2} \dots = 0$

№ 7.6(б)

$\oint\limits_{|z|=r}\frac{dz}{(z-a)^n(z-b)}$, $|a|<r<|b|$, $n\in\mathbb{N}$



$f^{(n)}(z)=\frac{n!}{2\pi i}\int_\gamma \frac{f(\xi)d\xi}{(\xi-z)^{n+1}}$

$f(z)=\frac{1}{z-b}\in A(|z|\le r)$

$\oint \frac{dz}{(z-a)^n(z-b)} = 2\pi i\left(\frac{1}{z-b}\right)^{(n-1)}\Big|_{z=a}\frac{1}{(n-1)!} =$

$\left(\frac{1}{z-b}\right)^{(k)} = (-1)^k\cdot k!\cdot\frac{1}{(z-b)^{k+1}}$

$= (-1)^{n-1}\frac{2\pi i}{(a-b)^n}$

$\int\limits_{|z|=r}\frac{dz}{(z-a)(z-b)} = \frac{2\pi i}{a-b}$, $\lim\limits_{r\to\infty}\int\frac{dz}{(z-a)(z-b)}=0$

$\frac{1}{(z-a)(z-b)} = \left(\frac{1}{z-a}-\frac{1}{z-b}\right)\cdot\frac{1}{a-b}$

Следствие из Теоремы Лиувилля

$f(z) \in A(\mathbb{C})$, $|f(z)| \le A$, A не зависит от z, $A>0$

$\Rightarrow f(z) = \mathrm{const}$. (α)

Если $|f(z)| \le A|z|^{\alpha}$, $|z| \ge z_0 > 0 \Rightarrow$

$\Rightarrow f(z)$ — многочлен степ. $\le [\alpha]$

$$(\alpha):\ \int\limits_{|z|=r} \frac{f(z)\,dz}{(z-a)(z-b)} = \left(\int\limits_{|z|=r} \frac{f(z)}{z-a}\,dz - \int\limits_{|z|=r} \frac{f(z)dz}{z-b}\right)\cdot\frac{1}{a-b} = \frac{1}{a-b}\left(f(a)-f(b)\right)$$

$$|I(\ast)| \le \frac{A\,2\pi r}{(r-|a|)(r-|b|)} \to 0 \ \Rightarrow f(a)=f(b)$$

γ — ~~замкн~~ гр. — замкн. кус.-гл. крив. Жордана

Если $f(z)$ анал. во внешн. обл. D и $\exists \lim\limits_{z\to\infty} f(z) = A \ne \infty$

$$\frac{1}{2\pi i}\oint_{\gamma} \frac{f(\xi)d\xi}{\xi - z} = \begin{cases} A - f(z), & z \notin \overline{D} \\ A, & z \in D \end{cases}$$

$$I = \frac{1}{2\pi i}\int\limits_{|\xi - z|=r} \frac{f(\xi)d\xi}{\xi - z} =$$

$\xi - z = re^{i\varphi}$

$$= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z+re^{i\varphi})\,\cancel{ire^{i\varphi}}\,d\varphi}{\cancel{re^{i\varphi}}} =$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(z+re^{i\varphi})\,d\varphi,\quad I - A = \frac{1}{2\pi}\int_0^{2\pi} f(z+re^{i\varphi})d\varphi - \frac{1}{2\pi}\int_0^{2\pi} A\,d\varphi = \frac{1}{2\pi}\int_0^{2\pi}\left(f(z+re^{i\varphi}) - A\right)d\varphi.$$

$\xrightarrow[z\to\infty]{} 0$

№7.15

~~$f(z)$ аналит. в $\{z\in\mathbb{C}: \dots$~~

$f(z) \in A(\overline{D})$ D — односвязн. область

$W_n(z) = (z-z_1)(z-z_2)\dots(z-z_n)$, $z_k \ne z_j$, $k\ne j$

$z_j \in D$, $j = \overline{1,n}$ D — ...

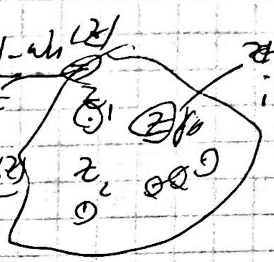
$$P(z) = \frac{1}{2\pi i}\oint_{\gamma} \frac{f(\xi)}{W_n(\xi)}\cdot\frac{W_n(\xi) - W_n(z)}{\xi - z}\,d\xi,\quad z \in D.$$

Интерполяционный многочлен Лагранжа.

1) Обводим z_i кружками

2) D_{γ_i, γ_0} это многочлен $\le n-1$

$$P(z) = \sum_{i=1}^{n}\frac{1}{2\pi i}\int_{\gamma_i} \frac{f(\xi)}{W_n(\xi)}\cdot\frac{W_n(\xi) - W_n(z)}{\xi - z}\,d\xi \ + \frac{1}{2\pi i}\int_{\gamma_0}\frac{f(\xi)}{W_n(\xi)}\cdot\frac{W_n(\xi) - W_n(z)}{\xi - z}$$



$$\frac{1}{2\pi i}\int_{\gamma_i} \frac{f(\xi)(W_n(\xi) - W_n(z))}{(\xi - z_1)\dots(\xi - z_i)\dots(\xi - z_n)(\xi - z)}\,d\xi$$

$$= f(z_i)\cdot\frac{W_n(z_i) - W_n(z)}{\underbrace{(z_i - z_1)\cdots(z_i - z_n)}_{\text{кроме}}(z_i - z)}$$

— Многочлен степени $\le n-1$

$$\frac{1}{2\pi i}\int_{\gamma_0} \frac{f(\xi)(W_n(\xi) - W_n(z))}{W_n(\xi)(\xi - z)}\,d\xi = 0$$

$$f(z) - P(z) = \frac{1}{2\pi i}\int_{\gamma} \frac{f(\xi)d\xi}{\xi - z} - \frac{1}{2\pi i}\int_{\gamma}\frac{f(\xi)(W_n(\xi) - W_n(z))}{W_n(\xi)(\xi - z)}\,d\xi =$$

$$= \frac{1}{2\pi i}\int_{\gamma} \frac{f(\xi)}{\xi - z}\left(1 - \frac{W_n(\xi) - W_n(z)}{W_n(\xi)}\right)d\xi =$$

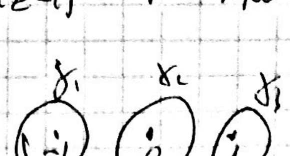
$$= \frac{1}{2\pi i}\int_{\gamma} \frac{f(\xi)}{\xi - z}\,\frac{W_n(z)}{W_n(\xi)}\,d\xi$$

Д/З ...

№7.5

$$\oint_{\gamma} \frac{dz}{z(z^2-1)}$$ по разл. замкн. кр. γ, не прох. через $0, \pm 1$

[illegible]



$$\int_{\gamma_1} \frac{dz}{z(z^2-1)} = \int_{\gamma_1} \frac{dz}{z(z-1)\cdot(z+1)} = \frac{2\pi i}{-1(-2)} = \frac{2\pi i}{2} = \pi i$$

$\int_{\gamma_2} \frac{}{} = -1 \cdot 2\pi i$, $\int_{\gamma_3} \frac{}{} = \frac{1}{2} 2\pi i = \pi i$

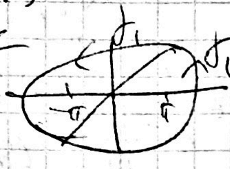
$\Sigma = 0$

№ 7.6 (1, 4, 9)

1) $\oint_{|z+i|=3} \frac{\sin z \, dz}{z+i} = 2\pi i \sin(-i) = \frac{e^{1} - e^{-1}}{2i} 2\pi i$

4) $\oint_{|z|=4} \frac{\cos z \, dz}{z^2 - \pi^2} = \oint_{|z|=4} \frac{\cos z}{(z-\pi)(z+\pi)} dz =$

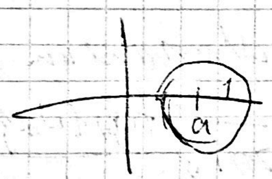
$f(z) = \frac{\cos z}{z - \pi}$



$= \int_{\gamma_1} + \int_{\gamma_2} \ge 0$

$2\pi i \frac{\cos(\pi)}{2\pi}$, $\frac{\cos(\pi) 2\pi i}{2\pi}$

9) $\oint_{|z-a|=1} \frac{e^z z \, dz}{(z-a)^3} = I$



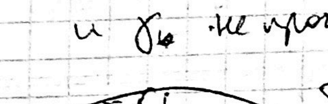
$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_{|z-a|=1} \frac{f(z) dz}{(z-a)^{n+1}}$

$I = \frac{2\pi i}{2!} (e^z \cdot z)'' |_a = \pi i e^a (a+2)$

№ 7.11

$\int_\gamma \frac{dz}{W_n(z)}$ $W_n(z) = (z - z_1) \cdots (z - z_n)$

и γ не прох. через z_i, z_i - разл.

Т.к.  т.к. симм. $\to 0$.

2) $\sum_{i=0}^{n} \frac{1}{\prod_{j=1}^{n} (z_i - z_j)} = 0$, $n \ge 2$

$\prod_{j=1}^{n} (z_i - z_j) = \frac{W(z)}{z - z_i}$, $W_n(z) = \prod_{i=1}^{n} (z - z_i)$

$\sum_{i=1}^{n} \frac{z - z_i}{W_n(z)} \Big|_{z = z_i} = \sum_{i=1}^{n} \frac{1}{2\pi i} \oint_{\gamma_i} \frac{d\xi}{W_n(\xi)} =$

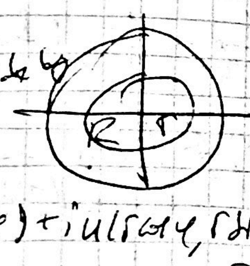
$= \frac{1}{2\pi i} \oint_\gamma \frac{1 \, d\xi}{W_n(\xi)} = 0$

аналог.

№ 7.24

$f(z)$ - ан. в $\{z : |z| < R\}$ и непр. в $\{z : |z| \le R\}$; $\iint_{r \le |z| \le R} f(z) dx dy$

$I = \iint_{r \le |z| \le R} u(x,y) + i v(x,y) \, dx dy$



$= \int_0^{2\pi} \int_r^R (u(r\cos\varphi, r\sin\varphi) + i u(r\cos\varphi, r\sin\varphi)) \, r \, dr \, d\varphi$

$\int_r^R r \, dr \int_0^{2\pi} u(re^{i\varphi}) + i v(re^{i\varphi}) \, d\varphi$

$\int_0^{2\pi} u(re^{i\varphi}) d\varphi = 2\pi u(0,0)$

$\int_0^{2\pi} v(re^{i\varphi}) d\varphi = 2\pi v(0,0)$

$\int_r^R 2\pi u(0,0) r \, dr + i \int_r^R 2\pi v(0,0) r \, dr =$

$= 2\pi (u(0,0) + i v(0,0)) \cdot \frac{r^2}{2} \Big|_r^R =$

$= \pi f(0) (R^2 - r^2)$

Кл. работа

$f(z) \in A\{z : |z| < R\} \cap C\{z : |z| \le \rho\}$

$f(z) = u(x, y) + iv(x, y)$

$$\iint\limits_{2\le|z|\le R} f(z)\,dx\,dy = \iint\limits_{2\le|z|\le R} u(x,y)\,dx\,dy + i\iint\limits_{2\le|z|\le R} v(x,y)\,dx\,dy =$$

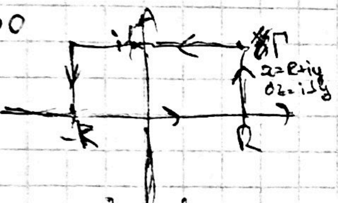
$$= \left|\begin{array}{l} x=\rho\cos\varphi \\ y=\rho\sin\varphi \\ J=\rho \\ \varphi\in[0,2\pi) \end{array}\right| = \int_2^R d\rho\int_0^{2\pi} u(\rho\cos\varphi,\rho\sin\varphi)\cdot\rho\,d\varphi + i\int_2^R d\rho\int_0^{2\pi} v(\rho\cos\varphi,\rho\sin\varphi)\rho\,d\varphi =$$

$$= 2\pi u(0,0)\int_2^R \rho\,d\rho + i2\pi v(0,0)\int_2^R \rho\,d\rho = (R^2-2^2)\pi f(0)$$

№ 6.49

$$I(b) = \int_0^{+\infty} e^{-x^2}\cos 2bx\,dx = \frac{1}{2}\int_{-\infty}^{+\infty} e^{-x^2}\cos 2bx\,dx =$$

1) $b>0$



$$= \frac{1}{2}\lim_{R\to+\infty}\int_{-R}^{R} e^{-x^2}\cos 2bx\,dx = \frac{1}{2}\lim_{R\to+\infty}\mathrm{Re}\int_{-R}^{R} e^{-x^2+2bix}\,dx$$

$f(z) = e^{-z^2}\cdot e^{2biz}$, аналит. $\Rightarrow \oint f(z)\,dz = 0 =$

$$= \int_{-R}^{R} e^{-x^2}\cdot e^{2bix}\,dx + \underbrace{\int_0^b e^{-(R^2+iy)^2}\cdot e^{2bi(R+iy)}\,dy}_{\xrightarrow[R\to+\infty]{}0} + \int_R^{-R} e^{-(x+ib)^2}\cdot e^{2bi(x+ib)}\,dx + i\underbrace{\int_b^0 e^{-(-R+iy)^2} e^{2ib(-R+iy)}\,dy}$$

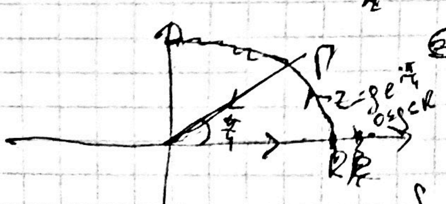
$$0 = \lim_{R\to+\infty}\int_{-R}^{R} e^{-x^2}\cdot e^{2bix}\,dx - \lim_{R\to+\infty}\int_{-R}^{R} e^{-x^2-2xib+b^2+2bix-2b^2}\,dx$$

$$\Rightarrow \lim_{R\to+\infty}\int_{-R}^{R} e^{-x^2}\cdot e^{2bix}\,dx = \sqrt{\pi}$$

$$\Rightarrow \frac{1}{2}\lim_{R\to+\infty}\mathrm{Re}\int_{-R}^{R} e^{-x^2}e^{2bix}\,dx = \frac{1}{2}e^{-b^2}\sqrt{\pi}$$

№ 6.50

$$I = \int_0^{+\infty}\sin x^2\,dx = \int_0^{+\infty}\cos x^2\,dx = \frac{\sqrt{\pi}}{2\sqrt{2}},\quad I = \lim_{R\to+\infty}\int_0^R \sin x^2\,dx =$$



$$= \lim_{R\to+\infty}\int_0^R e^{ix^2}\,dx$$

$f(z) = e^{iz^2}\in A(\mathbb{C})$

$$\oint f(z)\,dz = 0$$

$$0 = \int_0^R e^{ix^2}\,dx + \int_0^{\frac{\pi}{4}} e^{iR^2e^{2i\varphi}}\cdot Rie^{i\varphi}\,d\varphi + \int_R^0 e^{i\rho^2 e^{\frac{i\pi}{2}}}\cdot e^{i\frac{\pi}{4}}\,d\rho$$

$$\left|Ri\int_0^{\frac{\pi}{4}} e^{iR^2(\cos 2\varphi+i\sin 2\varphi)}e^{i\varphi}\,d\varphi\right| \le R\int_0^{\frac{\pi}{4}} e^{-R^2\sin 2\varphi}\,d\varphi,$$

$$\frac{2}{\pi}2\varphi \le \sin 2\varphi$$

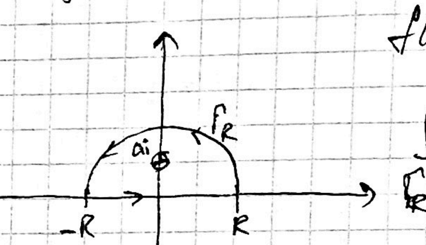
$$R\int_0^{\frac{\pi}{4}} e^{-R^2\frac{4}{\pi}\varphi}\,d\varphi = \left.\frac{Re^{-R^2\frac{4}{\pi}\varphi}}{-R^2\frac{4}{\pi}}\right|_0^{\frac{\pi}{4}} \xrightarrow[R\to+\infty]{} 0$$

$$e^{i\frac{\pi}{4}}\int_0^R e^{-\rho^2}\,d\rho \to e^{i\frac{\pi}{4}}\int_0^{+\infty} e^{-\rho^2}\,d\rho = e^{i\frac{\pi}{4}}\frac{\sqrt{\pi}}{2} = \frac{\sqrt{2}\sqrt{\pi}}{2\cdot 2} + i\frac{\sqrt{2}}{\sqrt{2}}\frac{\sqrt{\pi}}{2},\quad \int_0^{+\infty}\sin x^2\,dx = \frac{\sqrt{\pi}}{2\sqrt{2}},\quad \int_0^{+\infty}\cos x^2\,dx = \frac{\sqrt{\pi}}{2\sqrt{2}}$$

№ 6.55

$$I(a) = \int_0^{+\infty}\frac{\cos x}{x^2+a^2}\,dx = \frac{\pi}{2a}e^{-a}$$ — Интеграл Лапласа

$$f(z) = \frac{e^{iz}}{z^2+a^2}$$



$$\int_{\Gamma_R} f(z)\,dz = \int_{|\xi-ai|=\varepsilon} f(\xi)\,d\xi$$

$$I = \frac{1}{2}\int_{-\infty}^{+\infty}\frac{\cos x}{x^2+a^2}\,dx = \lim_{R\to\infty}\frac{1}{2}\int_{-R}^{R}\frac{\cos x}{x^2+a^2}\,dx = \frac{1}{2}\lim_{R\to+\infty}\mathrm{Re}\int_{-R}^{R}\frac{e^{ix}}{x^2+a^2}\,dx$$

$$\int_{|\xi-ai|=\varepsilon}\frac{e^{i\xi}}{\xi^2+a^2}\,d\xi = \frac{1}{2}\lim_{R\to+\infty}\int_{-R}^{R}\frac{e^{ix}}{x^2+a^2}\,dx$$

$$\int_{|\xi-ai|=\varepsilon}\frac{e^{i\xi}}{(\xi-ai)(\xi+ai)}\,d\xi = 2\pi i\frac{e^{-a}}{2ai} = \frac{\pi}{a}e^{-a}$$

№ 6.52

$$\int_0^{\infty}\frac{\sin^{2n+1}x}{x}\,dx,\ n\in\mathbb{N},\quad \sin^{2n+1}x = \left(\frac{e^{ix}-e^{-ix}}{2i}\right)^{2n+1} =$$

$$= \frac{\sum\limits^{2n+1} C_{2n+1}^k e^{kix}(-1)^{(2n+1)-k}\cdot e^{-(2n+1-k)ix}}{2^{2n+1}(-1)^n\cdot i} =$$

$$= \frac{\sum\limits_{k=0}^{2n+1} C_{2n+1}^k(-1)^{k+1}e^{ix(2k-2n-1)}}{2^{2n+1}(-1)^n\cdot i} = \frac{\sum\limits_{k=0}^{2n+1} C_{2n+1}^k(-1)^k e^{ix(2n+1-2k)}}{2^{(2n+1)}}$$

$$= \frac{\sum\limits_{k=0}^{2n+1} C_{2n+1}^k(-1)^k\sin[x(2n+1-2k)]}{2^{(2n+1)}(-1)^n} = \frac{2\sum\limits_{k=0}^{n} C_{2n+1}^k(-1)^k}{2^{(2n+1)}(-1)^n}$$

$\sin x[2n+1-2k]$,

$$\int_0^{+\infty} \frac{\sin^{(2n+1)} x}{x} dx = \frac{(-1)^n \sum_{k=0}^{n} C^k_{2n+1}(-1)^k \cdot \frac{\pi}{2}}{2^{2n}} =$$

$$= \frac{\pi}{2^{2n+1}} (-1)^n \sum_{k=0}^{n} C^k_{2n+1} (-1)^k$$

№6.58

$$\int_0^{+\infty} \frac{\cos^{2n}\alpha x - \cos^{2n}\beta x}{x} dx, \ \alpha\beta \neq 0$$

$\exists\ \alpha, \beta > 0$ — Формула Фруллани

$$\cos^{2n}\alpha x = \left(\frac{e^{i\alpha x} + e^{-i\alpha x}}{2}\right)^{2n} = \qquad \int_0^{+\infty} \frac{\cos\alpha x - \cos\beta x}{x} = \ln\frac{\beta}{\alpha}$$

$$= \frac{1}{2^{2n}} \sum_{k=0}^{2n} C^k_{2n} \cdot e^{i\alpha x k} \cdot e^{-i\alpha x(2n-k)} =$$

$$= \frac{1}{2^{2n}} \sum_{k=0}^{2n} C^k_{2n} e^{i\alpha x(2k-2n)} = \frac{1}{2^{2n}} \sum_{k=0}^{2n} C^k_{2n} \cos(\alpha x(2n-2k))$$

$$= \frac{1}{2^{2n}} \left(C^n_{2n} + 2\sum_{k=0}^{n-1} C^k_{2n} \cos(\alpha x(2n-4k))\right)$$

$$(\cos\beta x)^{2n} = \frac{1}{2^{2n}}\left(C^n_{2n} + 2\sum_{k=0}^{n-1} C^k_{2n} \cos(\beta x(2n-4k))\right)$$

$$I = \frac{1}{2^{2n-1}} \sum_{k=0}^{n-1} C^k_{2n} \ln\frac{\beta}{\alpha}$$

=)

Д/з

№6.58

$$\int_0^{+\infty} \frac{\sin^{2n}\alpha x - \sin^{2n}\beta x}{x} dx, \ \alpha\beta \neq 0$$

$$\sin\alpha x = \frac{e^{i\alpha x} - e^{-i\alpha x}}{2i}$$

№6.57

$$\int_0^{+\infty} \left(\frac{\sin x}{x}\right)^n dx, \ n \in \mathbb{N}$$

~~По зачету,~~.

№6.56

$a > 0, b > 0$

$$\int_0^{+\infty} \frac{\cos 2ax - \cos 2bx}{x^2} dx = \pi(b-a)$$

№6.60

$$\int_0^{+\infty} \frac{3x^2 - a^2}{(x^2+b^2)^2} \cos mx \, dx = \frac{\pi e^{-mb}}{4b^3} (3b^2 - a^2 - mb(3b^2+a^2))$$

$$f(z) = \frac{(3z^2 - a^2)}{(z^2+b^2)^2} e^{imz}$$

$$\oint_{|\xi - bi| = \varepsilon} \frac{(3\xi^2 - a^2) e^{im\xi}}{(\xi - bi)^2(\xi + bi)^2} d\xi = 2\pi i \cdot e^{-mb}\left(\frac{\quad}{\quad}\right)$$

№6.62

Аналог.

Упр. работа

№ 6.54

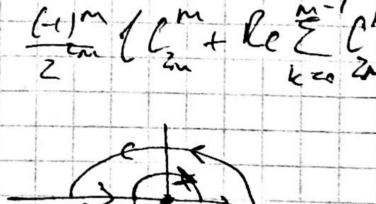
$$\int_0^{+\infty}\left(\frac{\sin x}{x}\right)^n dx, \ n\in\mathbb{N};$$

$$n=2m;\ \sin^{2m}x=\left(\frac{e^{ix}-e^{-ix}}{2i}\right)^{2m}=\frac{1}{2^{2m}(-1)^m}\sum_{k=0}^{2m}C_{2m}^k e^{ixk}e^{-ix(2m-k)}$$

$$=\frac{(-1)^m}{2^{2m}}\sum_{k=0}^{2m}C_{2m}^k e^{-ix(2m-2k)}(-1)^k$$

$$=\frac{(-1)^m}{2^{2m}}\left(C_{2m}^m+\sum_{k=0}^{m-1}C_{2m}^k\cos(x(2m-2k))(-1)^k\right)$$

$$=\frac{(-1)^m}{2^{2m}}\left(C_{2m}^m+\operatorname{Re}\sum_{k=0}^{m-1}C_{2m}^k e^{ix(2m-2k)}(-1)^k\right)$$



$$f(z)=\frac{(-1)^m}{2^{2m}}\left(C_{2m}^m+\operatorname{Re}\sum_{k=0}^{m-1}C_{2m}^k e^{iz(2m-2k)}(-1)^k\right)\cdot\frac{1}{z^{2m}}$$

$$I=\frac{1}{2}\lim_{\substack{R\to+\infty\\ r\to 0}}\left(\int_{-R}^{-r}\left(\frac{\sin x}{x}\right)^n dx+\int_r^R\left(\frac{\sin x}{x}\right)^n dx\right)$$

По лемме Жордана $\int$ по $\frown$ $=0$.

$$0=\lim_{\substack{R\to+\infty\\ r\to 0}}\left(\int_{-R}^{-r}\left(\frac{\sin x}{x}\right)^n dx+\int_r^R\left(\frac{\sin x}{x}\right)^n dx+\int_{C_r}f(z)dz\right.$$

$+ \int_{C_R} f(z)dz$, $\quad 2I = -\lim_{\substack{r\to 0\\ R\to\infty}} \int_{C_r} f(z)dz$

$$\int_{C_r}\left(\frac{(-1)^m}{2^{2m}}\, i\, C_{2m}^m + \mathrm{Re}\sum_{k=0}^{m-1} C_{2m}^k e^{iz(2m-2k)}(-1)^k\right)\frac{1}{z^{2m}}$$

$$\int_{C_r}\frac{dz}{z^{2m}} = \left| \begin{matrix} z=re^{i\varphi} \\ dz = rie^{i\varphi}d\varphi \end{matrix}\right| = \int_\pi^0 \frac{rie^{i\varphi}}{r^{2m}e^{i2m\varphi}}d\varphi =$$

$$= \frac{i}{r^{2m-1}}\int_\pi^0 e^{i\varphi(1-2m)}d\varphi = \frac{i(-1)^m}{r^{2m-1}} \dots = \frac{(-1)^m\, 2}{r^{2m-1}(1-2m)}$$

$e^z = \sum_{k=0}^\infty \frac{z^k}{k!}$, $a_{2m-1} = \frac{1}{(2m-1)!}$;

$$e^{iz(2m-2k)} \to \frac{1}{(2m-1)!}\,(i(2m-2k))^{2m-1} = \frac{2^{2m-1}(m-k)^{2m-1}}{(2m-1)!}(-1)^{m+1}\, i$$

$$2I = \frac{(-1)^m}{2^{2m}}\sum_{k=0}^{m-1} C_{2m}^k (-1)^{k+m}\, i\, \frac{1}{(2m-1)!}\, 2^{2m-1}(m-k)^{2m-1}\cdot \pi i$$

$$\int_0^\pi \frac{dz}{z} = \int_0^\pi \frac{ire^{i\varphi}}{re^{i\varphi}}d\varphi = \pi i$$

№ 8.2

$(-1)^{m-k}$

1) $\sum_{n=1}^\infty \frac{z^n}{n}$, $\quad R = \frac{1}{\overline{\lim\limits_{n\to\infty}}\sqrt[n]{|a_n|}} = \frac{1}{\lim\limits_{n\to\infty}\sqrt[n]{\frac{1}{n}}} = 1$

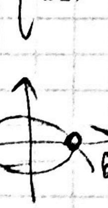
$f(z) = \sum_{n=1}^\infty \frac{z^n}{n}$, $|z|<1$, $|z|=1$, $z=e^{i\varphi}$

сх. $\forall\varphi\ne 2\pi k$

$$f(e^{i\varphi}) = \sum_{n=1}^\infty \frac{e^{in\varphi}}{n} = \sum_{n=1}^\infty\frac{\cos n\varphi}{n} + i\sum_{n=1}^\infty \frac{\sin n\varphi}{n}$$

$\sum_{n=1}^\infty \frac{\sin n\varphi}{n}$ $\forall\varphi$ сх. $\left|\sum_{k=1}^n \sin k\varphi\right|$ $\quad \sin\frac{\varphi}{2}$

$z_0 = 1$
$\varphi = 2\pi k$



все, кроме 0

Задача. Ряд сход. н. только внутри ~~...~~ на окруж. радиуса.

$\sum_{n=0}^\infty z^n$, $R=1$; $|z|=1$ $\not\to 0$

$f(z) = \sum_{n=0}^\infty z^n = \frac{1}{1-z}$, показать, что нет р.сх.

$\exists\varepsilon>0,\ \forall N\in\mathbb{N},\ \exists n\ge N: \exists z: |z|<1,$

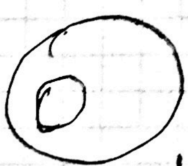
$\left|f(z) - \sum_{k=1}^n z^k\right| \ge \varepsilon$, $\quad \sum_{k=1}^n z^k = \frac{1-z^{n+1}}{1-z}$

$$f(z) - \frac{1-z^{n+1}}{1-z} = \frac{1}{1-z} - \dots = \frac{z^{n+1}}{1-z} \overset{z=1-\frac{1}{n}}{=} n\left(1-\frac{1}{n}\right)^{n+1} \xrightarrow[n\to\infty]{} +\infty$$

$z = 1-\frac{1}{n}$

Нет р.сх. $|z|<1$, но есть р.сх. внутри

$\exists K = \{z: |z-z_0|<2\} \subset \{z: |z|<1\}$



$\Rightarrow \exists \{z: |z|<z_1\} \supset K$,
$z_1<1$

$$\left|f(z) - \sum_{k=1}^n z^k\right| \le \frac{z_1^{n+1}}{1-z_1} < \varepsilon.$$

№ 8.21

$f(z) = \sum_{n=0}^\infty a_n z^n$ огр. $|z|<1 \Rightarrow \sum_{n=0}^\infty |a_n|^2$ сх.

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{|f(re^{i\varphi})|^2}{1}d\varphi \le C^2, \quad r<1$$

$$\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\varphi})|^2 d\varphi = \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\varphi})\cdot\overline{f(re^{i\varphi})}\,d\varphi =$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{n=0}^{+\infty}a_n r^n e^{in\varphi}\right)\left(\sum_{k=0}^{+\infty}\bar a_k r^k e^{-ik\varphi}\right)d\varphi \overset{\text{р.сх.}}{=}$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{k,n=0}^{+\infty} a_k\bar a_k\cdot r^{k+n}e^{i\varphi(n-k)}\right)d\varphi =$$

$$\sum_{k,n=0}^{+\infty}\frac{1}{2\pi}\int_0^{2\pi}(a_n\bar a_k\cdot r^{n+k}e^{i\varphi(n-k)})\,d\varphi =$$

$$= \sum_{n=0}^{+\infty}\frac{1}{2\pi}\int_0^{2\pi}|a_n|^2 r^{2n}d\varphi = \sum_{n=0}^{+\infty}|a_n|^2 r^{2n} \le C^2 \quad \forall r<1$$

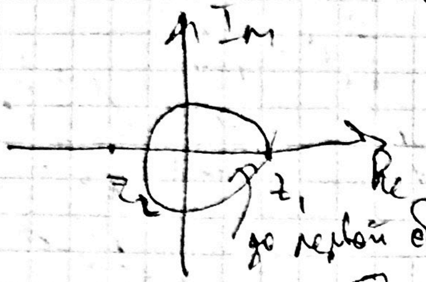
$$\sum_{n=0}^K |a_n|^2 r^{2n} \le C^2 \overset{r\to1}{\Rightarrow} \sum_{n=0}^K |a_n|^2 \le C^2 \Rightarrow \text{сх.}$$

№ 8.19

$f(z) = \frac{1}{1-z-z^2} = \sum_{n=0}^\infty a_n z^n$, $\quad a_n = a_{n-1}+a_{n-2}$

$A(z) = \frac{1}{1-z-z^2} = \sum_{n=0}^{+\infty} a_n z^n$, $\quad a_n = a_{n-1}+a_{n-2}$)

$1-z-z^2=0 \Rightarrow z_1=\frac{\sqrt5-1}{2},\ z_2=\frac{-\sqrt5-1}{2}$



$R=\frac{\sqrt5-1}{2}$

$f(z)=\frac{-1}{(z-z_1)(z-z_2)}=\frac{A}{z-z_1}+\frac{B}{z-z_2}$

$-1=A(z-z_2)+B(z-z_1)$

$A=-\frac{1}{z_1-z_2}\quad B=\frac{-1}{z_2-z_1}$

$\frac{1}{z-z_1}=\frac{1}{-z_1(1-\frac{z}{z_1})}=\frac{1}{-z_1}\left(1+\frac{z}{z_1}+\frac{z^2}{z_1^2}+\dots\right)\quad |z|<|z_1|$

$\frac{1}{z-z_2}=-\frac{1}{z_2}\sum_{k=0}^{\infty}\left(\frac{z}{z_2}\right)^k,\ |z|<|z_2|$

$a_n=\frac{1}{z_2-z_1}\left(-\frac{1}{z_1^{n+1}}\right)+\frac{1}{z_1-z_2}\left(-\frac{1}{z_2^{n+1}}\right)=$

$=$

№8.2 (10, 11)

10) $\sum_{n=0}^{\infty}\cos(in)z^n\quad R=\frac{1}{\lim_{n\to\infty}\sqrt[n]{|\cos(in)|}}=$

$=\frac{1}{\lim_{n\to\infty}\sqrt[n]{\frac{e^{-n}+e^{n}}{2}}}=\frac{1}{e}$

$\approx\sqrt[n]{\frac{e^n}{2}}=\frac{e}{2^{n}}\to e$

11) $\sum_{n=0}^{\infty}\frac{(n+\alpha^n)^n}{n!}z^n\quad \frac{1}{\lim_{n\to\infty}\frac{n+\alpha^n}{\sqrt[n]{n!}}}=+\infty$

$\frac{\sum_{k=0}^{n}C_n^k n^k\alpha^{n(n-k)}}{n!}$

№8.3 (3)

$\sum_{n=0}^{\infty}[3+(-1)^n]^n z^n$

$R=\frac{1}{\overline{\lim}\sqrt[n]{|3+(-1)^n]^n|}}=\frac{1}{4}$

$\sum_{n=0}^{\infty}[3+(-1)^n]^n z^n$

№8.11 (1, 2)

1) $\sum_{n=1}^{\infty}\frac{\cos n\varphi}{n}=-\ln\left|2\sin\frac{\varphi}{2}\right|,\ 0<\varphi\le\pi$

2) $\sum_{n=1}^{\infty}\frac{\sin n\varphi}{n}=\frac{\pi-\varphi}{2}\quad 0<\varphi<2\pi$

№8.17(1,3)

1) $(1-z)^{-2}$ по ст. $(z-z_0)$, $|z_0|>1$

$f(z)=\frac{1}{(1-z)^2}$; $f'(z)=+2\frac{1}{(1-z)^3}$

$f''(z)=\frac{2\cdot3}{(1-z)^4}$

$f^{(n)}(z)=\frac{(n+1)!}{(1-z)^{n+2}}$

$\Rightarrow (1-z)^{-2}=\sum\limits_{n=0}\frac{(n+1)!}{(1-z_0)^{n+2}\,n!}(z-z_0)^n$

$=\sum\limits_{n=0}^{\infty}\frac{n+1}{(1-z_0)^{n+2}}(z-z_0)^n.$

3) $(z^2+1)^{-1}=\frac{1}{1+z^2}=f(z)$

$f'(z)=-\frac{2z}{(1+z^2)^2}$; $f''(z)=-2\left(\frac{(1+z^2)^2-z2(1+z^2)\cdot2z}{(1+z^2)^4}\right)$

$1+z^2=-i^2+z^2\ldots$

$1+z^4+2z^2-4z^2-4z^6$

$=(1+2z^2+z^4-4z^6)$

№8.20(а,б)

а) $f(z)=\sum\limits_{n=0}^{\infty}a_nz^n$, $|z|<R$

а) $0<r<R$ нер-во Коши в рав-во

$\Rightarrow f(z)=a_kz^k$

б) $\frac{1}{2\pi}\int_0^{2\pi}$

№8.22

$f(z)=\sum\limits_{n=0}^{\infty}a_nz^n$, $|z|<1$ $\quad\sum|a_n|^2$ сх.

$\frac{1}{2\pi}\int_0^{2\pi}|f(r,\varphi)-f(\rho,\varphi)|^2d\varphi\Leftrightarrow$

$\Leftrightarrow\sum\limits_{n=0}^{\infty}|a_n|^2(r^n-\rho^n)^2$, $r,\rho<1$.

Кн.-р

№8.14(1,3,8)

3) $\frac{1}{a^2+z^2}=f(z)$, $f'(z)=-\frac{2z}{a^2+z^2}$

$f''(z)=\frac{-2(a^2+z^2-z(2z))}{(a^2+z^2)^2}=\frac{-2(a^2-z^2)}{(a^2+z^2)^2}$

$f(z)=\frac{1}{z^2-(ia)^2}\overset{**}{=}\left(\frac{1}{z-ia}+\overset{*}{\frac{1}{z+ia}}\right)\frac{1}{2ia}$

$-\frac{1}{ia}\frac{1}{1-\frac{z}{ai}}=-\frac{1}{ia}\Big($

$1+\frac{z}{ai}+\frac{z^2}{(ai)^2}+\ldots\Big)$,

$*=\frac{1}{ia}\left(\frac{1}{1+\frac{z}{ia}}\right)=\frac{1}{ia}\left(1-\frac{z}{ia}+\frac{z^2}{(ia)^2}+\ldots\right)$

2) $**=\frac{1}{2ia}\cdot\frac{1}{ia}\left(-1-\frac{z}{ai}-\frac{z^2}{(ai)^2}-\ldots-1+\frac{z}{ia}-\frac{z^2}{(ia)^2}+\ldots\right)$

$=+\frac{1}{2a^2}\cancel{2}\left(1+\frac{z^2}{(ai)^2}+\frac{z^4}{(ai)^4}+\ldots\right)=$

$\cancel{\sum\frac{1}{a^2}\left(1-\frac{z^2}{a^2}\right.}$ $R=\frac{1}{\lim\limits_{n\to\infty}\sqrt[n]{|a_n|}}=\frac{1}{a}$

№8.15

№8.16

11) $\int_0^z \frac{\sin\xi}{\xi}\,d\xi = f(z)$

... разложим эти

№8.18(1,2)

2) $e^{z\sin z} = e^{(t+i)\sin(t+i)}$ ... разл. по Маклорену

№8.19

№8.26(1)

$f(0)=1,\ f'(z)=f(z) \Rightarrow f''(z)=f'(z)$

$f(z) = \sum_{k=0}^{\infty} \frac{1}{k!} z^k$

№8.47(1,2,3),4)

Аналог.

Д/р

№8.16(7,8,9,10)

8) $e^z\cos z = (1+z+\frac{z^2}{2!}+\ldots)(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\ldots)$

№8.17(2)

2) $(z^2-1)^{-1} = $ ___ сумм дробей

№8.18(3,4)

3) $\frac{z^2}{(1+z)^2} = \frac{(t+i)^2}{(1+i+t)^2} = \frac{(i+t)^2}{(1+i+t)(1+i+t)}$

№8.20

–

№8.26(2)

___

№ 8.48

... Максимум.

№

кр.

№ 9.16(1,2,3)

1) $\sum_{n=-\infty}^{\infty} 2^{-|n|} z^n$ | $R = \frac{1}{\lim_{n\to\infty} \sqrt[n]{|2^{-|n|}|}} =$

$= \frac{1}{\lim_{n\to\infty} 2^{-1}} = 2$; $r = \lim_{n\to\infty} \sqrt[n]{2^{-n}} = \frac{1}{2}$

2)

3)

№ 9.17(1,3,6) ?

1) $f(z) = \cos(1/z)$, $z = 0$

№ 9.21

$e^{i(z-1/z)/2}$, $\{0<|z|<\infty\}$ ряд. в обл-ти ?

~~1.2 (8,9)~~, 1.4(9,10),
1.5 ~~(1,4,5)~~, ~~1.10(1,2,3)~~

~~3.46, 3.38, 3.34,~~
~~3.33, 3.32~~

~~6.28, 6.22, 6.16, 6.10,~~
~~6.6, 5.1 (6,7,9)~~

~~7.5, 7.6, 7.4 (1,7,9), 7.20,~~
~~7.11, 7.24~~

~~6.57, 6.54, 6.56,~~
6, ~~60, 6.62~~

8.2 (1a, б), 8.22,
8.17 (1,2), 8.11 (1,2), ?
~~8.2 (1a, 1)~~, 8.3(3)
?

~~8.14 (1,3,7)~~ 8.15, ~~8.16 (1,2,10)~~ ~~8.17 (1,2)~~, 8.15, ~~8.26 (1)~~ ~~8.47 (1,2,3,4)~~	~~8.14 (2,4,5,6)~~ 8.16 (7,8,9,10) ~~8.17 (2)~~ 8.11 (3,4) 8.20 ~~8.26 (2)~~ ~~7.48~~
~~8.16 (1,2,3)~~ 9.17 (1,3,6) ? 8.21 ? 9.23 (1,2) ? 9.24 ? 9.25 (1-5) 9.26 (1) ?	8.16 (4-7) 9.17 (2,4,5,7) 9.18 9.19 9.23 (3,4) 9.25 (6-10) 9.26 (3,4)

9.16, 20, 21, 22 наперед, если e^z определ. как
$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}, \text{Эйлер}$$

тетрадь/48 листов

Тетрадь Кузнецова М.К. 202 группа ВМК

Методич. рук-ва по мат. анализу II курс 3,4 семестр Леонтьева, Домрина

Собственные интегралы, зависящие от параметров

$F(y)=\int_a^b f(x,y)\,dx,\ y\in Y$

Пусть $y\in[c,d]$

Теор.1 ∃ $f(x,y)\in C([a,b]\times[c,d]) \Rightarrow F(y)\in C([c,d])$

Теор.2 ∃ $f(x,y)\in C([a,b]\times[c,d])$, $\exists f'_y(x,y)\in C([a,b]\times[c,d]) \Rightarrow \exists F'(y)\in C([c,d])$, $F'(y)=\int_a^b f'_y(x,y)\,dx$

$F(y)=\int_{\alpha(y)}^{\beta(y)} f(x,y)\,dx,\ y\in[c,d],\ \alpha(y),\beta(y)\in[a,b]$

=) Теор.1' ∃ $f(x,y)\in C(\_\_,\ \alpha(y),\beta(y)\in C([c,d])) \Rightarrow F(y)\in C([c,d])$

Теор.2' ∃ $f(x,y)\in C(\_\_,\_\_)$, $\exists f'_y(x,y)\in C([a,b]\times[c,d])$, $\exists \alpha'(y),\beta'(y)\in C([c,d]) \Rightarrow$

$\Rightarrow \exists F'(y)=\int_{\alpha(y)}^{\beta(y)} f'_y(x,y) + \beta'(y)\cdot f(\beta(y),y)-\alpha'(y)\cdot f(\alpha(y),y)$

№3612

$F(y)=\int_0^1 \frac{y f(x)}{x^2+y^2}\,dx$, $f(x)\in C([0,1])$, $f(x)>0$ на $[0,1]$. $\to \exists m>0: f(x)\geq m$

$F(0)=0,\ y=0;\ y\neq0 \Rightarrow F(y)\in C(\mathbb{R}\setminus\{0\})$

$y>0: \int_0^1 \frac{yf(x)}{x^2+y^2}\,dx \geq m\int_0^1\frac{y}{x^2+y^2}\,dx = m\,\mathrm{arctg}\frac{x}{y}\Big|_0^1 = m\,\mathrm{arctg}\frac1y \Rightarrow$

$\Rightarrow F(y)\geq m\,\mathrm{arctg}\frac1y \Rightarrow \cancel{\sup_{y>0} F(y)}\ \varliminf_{y\to+0} F(y) \geq m \lim_{y\to0+}\mathrm{arctg}\frac1y = m\frac{\pi}{2}$

Если $F(y)$ в т. $y=0$ была непрер., то $F(0)=0 \neq \varliminf_{y\to+0} F(y)$, но $0\neq m\frac{\pi}{2}$ =)

$F(y)$ в т. 0 имеет разрыв.

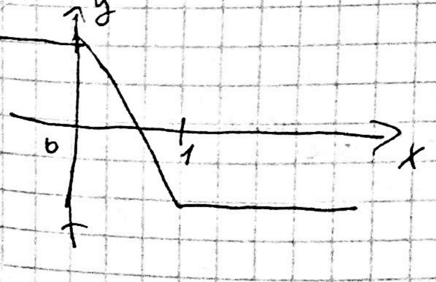
Задача: разрыв 1го или 2го рода.

№3711

$F(y)=\int_0^1 f(x,y)\,dx$, $f(x,y)=\mathrm{sgn}(x-y)$

$y>1$: $F(y)=-1$ | $0<y<1$: $F(y)=\int_0^y \underbrace{f(x,y)}_{-1}dx+\int_y^1\underbrace{f(x,y)}_{1}dx = -y+(1-y)=1-2y$

$y<0$: $F(y)=1$



№3713(б)

№3713(б)

б) $\lim\limits_{\alpha\to 0}\int_0^2 x^2\cos\alpha x\,dx$

$F(\alpha)=\int_0^2 x^2\cos\alpha x\,dx$ - непрер. на пл. $\Rightarrow F(\alpha)$ непрер. $\Rightarrow \lim\limits_{\alpha\to 0} \text{—} = F(0) = \int_0^2 x^2\,dx =$

$= \frac{8}{3}$

2) $\lim\limits_{n\to\infty}\int_0^1 \frac{dx}{1+\left(1+\frac{x}{n}\right)^n} =$, $F(x)=\int_0^1\frac{dx}{1+\left(1+\frac{x}{n}\right)^n}$ непрер. $x\in\mathbb{R}$, $n>20$ $\Rightarrow = \int_0^1\frac{dx}{1+e^x} = \int_0^1\frac{e^{-x}dx}{e^{-x}+1} =$ $1-\ln(1+e)+\ln 2$.

№3714

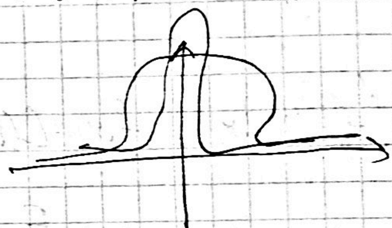
1) $f(x)\in C([A,B])$ $\quad \cancel{F(h)=\int_a^x \frac{f(t+h)-f(t)}{h}dt}$

$\exists F(x)$ - первообр. для $f(x)$ $\Rightarrow \lim\limits_{h\to 0}\frac{1}{h}\left(F(t+h)-F(t)\right)\Big|_a^x = \lim\limits_{h\to 0}\left(\frac{1}{h}\left(F(x+h)-F(x)\right)-\right.$

$\left.-\frac{1}{h}\left(F(a+h)-F(a)\right)\right) = \cancel{\lim\limits_{h\to 0}\frac{F(x+h)-F(x)}{dx}}\ f(x)-f(a).$

2) $\varphi_n(x)\ge 0$ на $[-1,1]$, $n\in\mathbb{N}$, 2) $\varphi_n(x)\rightrightarrows 0$ $n\to\infty$ на $0<\varepsilon\le|x|\le 1$

3) $\int_{-1}^1\varphi_n(x)dx\to 1$, $n\to\infty$.



$\int_{-1}^1 f(x)g(x)dx$ - непрер. лин. функ-л.

$\int_{-1}^1 f(x)\cdot\tilde{g}(x)\,dx = \tilde{g}(0)$

Реш-е: $\int_{-1}^1 f(0)\cdot\varphi_n(x)dx = f(0)\int_{-1}^1\varphi_n(x)dx \xrightarrow[n\to\infty]{} f(0)$

$\int_{-1}^1 f(x)\varphi_n(x)dx - \int_{-1}^1 f(0)\varphi_n(x)\,dx = \int_{-1}^1 (f(x)-f(0))\varphi_n(x)dx = \int_{-1}^{-\varepsilon}\text{—}\,dx + \int_{-\varepsilon}^{\varepsilon}\text{—}\,dx + \int_{\varepsilon}^{1}\text{—}\,dx$

$|f(x)|\le M$, $x\in[-1,1]$; $\forall\delta>0\ \exists\varepsilon>0$: $|f(x)-f(0)|<\delta$ $|x|<\varepsilon$

$\left|\int_{-\varepsilon}^{\varepsilon}\text{—}\,dx\right| \le \delta\cdot\int_{-\varepsilon}^{\varepsilon}\varphi_n(x)dx \le \delta\cdot\int_{-1}^1\varphi_n(x)dx \le 2\delta$

$\left|\int_{\varepsilon}^1\text{—}\,dx\right| \le 2M\int_\varepsilon^1\varphi_n(x)dx \le \delta$ $\quad \forall n>n_1,\ n_1>n_0$

$|A_n|\le 4\delta\quad \forall n>n_1$

№3718(а)

$F(\alpha)=\int_{\sin\alpha}^{\cos\alpha} e^{\alpha\sqrt{1-x^2}}dx$, $F'(\alpha)=\int_{\sin\alpha}^{\cos\alpha}\sqrt{1-x^2}\,e^{\alpha\sqrt{1-x^2}}dx - \sin\alpha\, e^{\alpha|\sin\alpha|} - \cos\alpha\, e^{\alpha|\cos\alpha|}$

$\int e^{x^2}dx$

$x=\sin t$

$dx=\cos t\,dt$

24

№ 3713(g)

$\lim_{R\to\infty} \int_0^{\frac{\pi}{2}} e^{-R\sin\theta} d\theta$; $I(R) = \int_0^{\frac{\pi}{2}} \underbrace{e^{-R\sin x}}_{\text{непр.}} dx \to I = \int_0^{\frac{\pi}{2}} 0\,dx = 0.$

№ 3715

$\lim_{y\to 0} \int_0^1 \frac{x}{y^2} e^{-\frac{x^2}{y^2}} dx = -\frac{1}{2}\int_0^1 e^{-\frac{x^2}{y^2}} d\left(-\frac{x^2}{y^2}\right) = -\frac{1}{2} e^{-\frac{x^2}{y^2}}\Big|_0^1 = -\frac{1}{2}\left(e^{-\frac{1}{y^2}} - 1\right) \Rightarrow$

$\Rightarrow \lim_{y\to 0} = \frac{1}{2}.$

$\lim_{y\to 0} \frac{x}{y^2} e^{-\frac{x^2}{y^2}} = \lim_{y\to 0} \frac{\frac{x}{y^2}}{e^{\frac{x^2}{y^2}}} = \lim_{y\to 0} \frac{-2\frac{x}{y^3}}{e^{\frac{x^2}{y^2}} \cdot x^2 \cdot (-2)\frac{1}{y^3}} = \lim_{y\to 0} e^{-\frac{x^2}{y^2}} x^{-1} = 0 \Rightarrow \frac{1}{2} \neq 0$ (!) нет.

№ 3716

$F(y) = \int_0^1 \ln\sqrt{x^2+y^2}\,dx$, $\ln\sqrt{x^2+y^2}\Big|_{(0,0)} = -\infty$ $B_\varepsilon(0,0)$ непр. ; $f'_y(x,y) = \frac{1}{\sqrt{x^2+y^2}} \cdot \frac{1}{2}\frac{1}{\sqrt{x^2+y^2}} \cdot 2y =$

$= \frac{y}{(x^2+y^2)}$

Нет.

№ 3718(2)

$F(\alpha) = \int_0^\alpha f(x+\alpha, x-\alpha)\,dx$ $\Big|\ \frac{d}{dy}\int_{a(y)}^{b(y)} f(x,y)\,dx = \int_a^b f'_y(x,y)\,dx + b' f(b,y) - a' f(a,y)$

$F(y) = \int_0^y f(x+y, x-y)\,dx$

$\cancel{F'(y) = \int_0^y f'_{(x+y)(x-y)}(x+y, x-y)\,dx + y\cdot f(2y,0)}$ $\quad F'(\alpha) = f(2\alpha,0) + \int_0^\alpha (f'_u(u,v) - f'_v(u,v))\,dx$

$= f(2\alpha,0) + 2\int_0^\alpha f'_u(u,v)\,dx - \int_0^\alpha (f'_u(u,v) + f'_v(u,v))\,dx = f(2\alpha,0) + 2\int_0^\alpha f'_u(u,v)\,dx -$

$- \int_0^\alpha \frac{d}{dx} f(u,v)\,dx = f(2\alpha,0) + 2\int_0^\alpha f'_u(u,v)\,dx - \underbrace{f(x+\alpha, x-\alpha)\Big|_{x=0}^{x=\alpha}}_{-(f(2\alpha,0) - f(\alpha,-\alpha))} = f(\alpha,-\alpha) + 2\int_0^\alpha f'_u(u,v)\,dx$

№ 3726

$J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(n\varphi - x\sin\varphi)\,d\varphi$ $n \in \mathbb{Z}$ удов. $x^2 J''_n(x) + x J'_n(x) + (x^2 - n^2) J_n(x) = 0$

$J'_n(x) = \frac{-1}{\pi}\int_0^\pi \sin\varphi \cdot \sin(n\varphi - x\sin\varphi)\,d\varphi$, $J''_n(x) =$ ____ $\Rightarrow$ ____ .

№ 3728

$u(x) = \int_0^1 K(x,y)\,v(y)\,dy$, $K(x,y) = \begin{cases} x(1-y), & x \le y \\ y(1-x), & x > y \end{cases}$ $v(y)$ непр. удов. $u''(x) = -v(x)$ $(0 \le x \le 1)$

$x = y$: $K(x,y) = x(1-y) = x(1-x)$

$y(1-x) \xrightarrow[x\to y]{} y(1-y) = x(1-x)$ непр.

отк $u(x) = \int_0^x y(1-x)v(y)\,dy + \int_x^1 x(1-y)v(y)\,dy \Rightarrow u'(x) = x(1-x)v(x) - \int_0^x y\,v(y)\,dy - x(1-x)v(x) +$

$+ \int_x^1 (1-y)v(y)\,dy = -\int_0^x y\,v(y)\,dy + \int_x^1 (1-y)v(y)\,dy$, $\Rightarrow u''(x) = -v(x)$.

Кн. р.

№ 3732

$$I(a,b) = \int_0^{\pi/2} \ln(a^2\sin^2 x + b^2\cos^2 x)\,dx = \int_0^{\pi/2} \ln\left(\frac{a^2+b^2}{2} + \frac{1}{2}\cos 2x(b^2-c^2)\right)dx \quad a^2+b^2 \neq 0$$

$$\frac{2a^2\sin x\cos x - 2b^2\sin x\cos x}{a^2\sin^2 x + b^2\cos^2 x} = \frac{2\sin x\cos x(a^2-b^2)}{a^2\sin^2 x + b^2\cos^2 x} = \frac{2\,tg\,x(a^2-b^2)}{b^2+a^2 tg^2 x}$$

$$\int_0^{\pi/2} \frac{2a\sin^2 x}{a^2\sin^2 x + b^2\cos^2 x}dx = \left|\begin{matrix} t = tg\,x \\ dx = \frac{dt}{1+t^2}\end{matrix}\right| = 2a\int_0^{+\infty} \frac{t^2}{a^2t^2+b^2}\frac{dt}{(1+t^2)} = \left|t^2 = u\right| = 2a\int_0^{+\infty}\frac{u}{a^2u+b^2}\cdot\frac{1}{1+u}dt$$

Берём произв-но по a.

$$= \frac{1}{a^2u+b^2}\left(\ldots - \frac{1}{1+u}\right) = \frac{1}{a^2u+b^2} - \frac{1}{a^2u+b^2}\cdot\frac{1}{1+u} \quad \frac{A}{a^2u+b^2} + \frac{B}{1+u}$$

$$\Rightarrow 2a\left(\int_0^{+\infty}\frac{1}{a^2t^2+b^2}\left(-\frac{b^2}{a^2-b^2}\right)dt + \int_0^{+\infty}\frac{1}{1+t^2}\cdot\frac{dt}{a^2-b^2}\right) = \frac{\pi}{a+b} \Rightarrow I(a,b) = \pi\ln(|a+b|) + C(b)$$

$a>0, b>0$

$a=b \Rightarrow I(b,b) = \int_0^{\frac{\pi}{2}} \ln(b^2)dx = \frac{\pi}{2}\ln(b^2) = \pi\ln(2b) + C(b) \Rightarrow C(b) = \pi(-\ln 2).$

$I(a,b) = \pi\ln(a+b) - \pi\ln 2$, Ответ: $\pi\ln\left(\frac{a+b}{2}\right)$

№ 3734

$$I(a) = \int_0^{\pi/2}\frac{arctg(a\,tg\,x)}{tg\,x}dx; \quad I'(a) = \int_0^{\frac{\pi}{2}}\frac{\frac{tg\,x}{1+(a\,tg\,x)^2}}{tg\,x}dx = \int_0^{\pi/2}\frac{dx}{1+a^2tg^2x} = \left|\begin{matrix} tg\,x = t \\ x = arctg\,t \\ dx = \frac{1}{1+t^2}dt\end{matrix}\right| \Rightarrow$$

$$\Rightarrow = \int_0^{+\infty}\frac{1}{1+a^2t^2}\frac{1}{1+t^2}dt = \int_0^{+\infty}\frac{dt}{1+t^2} - \int_0^{+\infty}\frac{dt}{1+a^2t^2}$$

$$\frac{1}{1+a^2u}\frac{1}{1+u} = \frac{A}{1+u} + \frac{B}{1+a^2u} \quad (1-a^2)A = 1, \quad A = \frac{1}{1-a^2}; \; B = \quad I = sign\,a\ln(1+|a|)$$

№ 3737

$$\int_0^1\frac{x^b - x^a}{\ln x}dx \quad (a>0, b>0); \quad I(a,b) = \int_0^1\frac{e^{\ln x}(e^b - e^a)}{\ln x}dx; \quad I(a,b) = \int_0^1 dx\int_a^b x^y dy =$$

$$= \int_a^b dy\int_0^1 x^y dx = \int_a^b dy\cdot\left.\frac{x^{y+1}}{y+1}\right|_0^1 = \int_a^b\frac{dy}{y+1} = \ln|y+1|\Big|_a^b = \ln\left|\frac{b+1}{a+1}\right|$$

№ 3738 (a)

$$\int_0^1 \sin\ln\left(\frac{1}{x}\right)\frac{x^b - x^a}{\ln x}dx = \int_0^1$$

$$\int_a^b x^y dy \quad |(\ln\ln x^a)' = \frac{1}{\ln x^a}\cdot\frac{1}{x^a}\cdot a x^{a-1}$$

$$= \int_0^1\sin(\ln\tfrac{1}{x})\int_a^b x^y dy = \int_a^b dy\int_0^1\sin(\ln\tfrac{1}{x})x^y dx; \quad I = \int_0^1\sin(\ln\tfrac{1}{x})x^y dy = \left|t = \ln\tfrac{1}{x}\right| = \int_0^{+\infty}\sin t\cdot e^{-t(y+1)}$$

... по частям. $= \frac{(1+y)^2+1-1}{1+(1+y)^2}dy \Rightarrow \int$

$\mathcal{D}/p.$

3741

$\int_0^{+\infty} \frac{e^{-ax}}{1+x^2}dx$ $a \geq 0$: $O^*\left(\frac{1}{x^{\varepsilon}}\right)_{\varepsilon > 1} \Rightarrow$ сх. $\quad a \geq 0$ $\frac{e^{-ax}}{1+x^2} \leq \frac{1}{1+x^2}$ $\int_0^{+\infty}$ — сх. $\Rightarrow$ сх.

$a<0$: $\lim\limits_{A\to+\infty} \int_0^A \frac{e^{-ax}}{1+x^2}dx$ не $\exists$ к.ч.

т.к. $\frac{e^{-ax}}{1+x^2} > \frac{1}{x}$ с нек. $x \Rightarrow$ —

3733

$\frac{d}{da}\int_0^{\pi} \ln(1-2a\cos x+a^2)dx = \int_0^{\pi} \frac{-2\cos x+2a}{1-2a\cos x+a^2}dx = \frac{1}{a}\int\left(1+\frac{a^2-1}{1-2a\cos x+a^2}\right)dx = \frac{\pi}{a} - \frac{1-a^2}{a}\int_0^{\pi}$

$\frac{dx}{(1+a^2)-2a\cos x} = \frac{\pi}{a} - \frac{1-a^2}{a(1+a^2)}\int_0^{\pi} \frac{dx}{1+\left(\frac{-2a}{1+a^2}\right)\cos x} = \frac{\pi}{a} - \frac{2}{a}\operatorname{arctg}\left(\frac{1+a}{1-a}\operatorname{tg}\frac{x}{2}\right)\Big|_0^{\pi} = \frac{\pi}{a} - \frac{2}{a}\frac{\pi}{2} = 0$

$\Rightarrow |a|<1$; $I(a) = C(a)$, $I(0) = 0 \Rightarrow C = 0 \Rightarrow I(a) = 0$, $|a|>1$ $b = \frac{1}{a}$ $|b|<1$ $I(b)=0$.

$I(a) = \int_0^{\pi} \ln\left(\frac{b^2-2b\cos x+1}{b^2}\right)dx = I(b) - 2\pi\ln|b| = -2\pi\ln|b| = 2\pi\ln|a|$

$|a| = 1 \Rightarrow \int_0^{\pi} \ln 2(1-\cos x)dx = \int_0^{\pi}\left(\ln 4 + 2\ln\sin\frac{x}{2}\right)dx = 2\pi\ln 2 + 4\int_0^{\frac{\pi}{2}}\ln\sin t\,dt =$

$I(1) \qquad = 2\pi\ln 2 + 4\left(-\frac{\pi}{2}\ln 2\right) = 0$, $I(-1) = 0 \Rightarrow I = \begin{cases} 0, & |a| \leq 1 \\ 2\pi\ln|a|, & |a| \geq 1 \end{cases}$

3735

$\int_0^{\frac{\pi}{2}} \ln\frac{1+a\cos x}{1-a\cos x}\frac{dx}{\cos x}$ $(|a|<1)$ $\quad \frac{1+a\cos x}{1-a\cos x} = \frac{1-a^2\cos^2 x}{1-2a\cos x+a^2\cos^2 x} \geq \frac{1-a^2}{1+2|a|+a^2} = \frac{1-a^2}{(1+|a|)^2} > 0$

$\lim\limits_{x\to\frac{\pi}{2}-0} \frac{1}{\cos x}\cdot\ln\frac{1+a\cos x}{1-a\cos x} = \lim\limits_{t\to 0}\frac{\ln(1+at)-\ln(1-at)}{t} = 2a$

$I'(a) = \int_0^{\frac{\pi}{2}}\left(\frac{1}{1+a\cos x} + \frac{1}{1-a\cos x}\right)dx = \frac{2}{\sqrt{1-a^2}}\operatorname{arctg}\left(\sqrt{\frac{1-a}{1+a}}\operatorname{tg}\frac{x}{2}\right)\Big|_0^{\frac{\pi}{2}} = \frac{\pi}{\sqrt{1-a^2}}$

$I(a) = \pi\arcsin a + C$ $(|a|<1)$, $I(0) = 0$, $C = 0$

3736

$\frac{\operatorname{arctg} x}{x} = \int_0^1 \frac{dy}{1+x^2y^2} \to \int_0^1 \frac{\operatorname{arctg} x}{x}\frac{dx}{\sqrt{1-x^2}} = \int_0^1 \int_0^1 \frac{dy}{1+x^2y^2}\frac{dx}{\sqrt{1-x^2}} \ominus$

$\int_0^1 \frac{dy}{1+x^2y^2}$

$\ominus \int_0^1 dy \int_0^1 \frac{dx}{(1+x^2y^2)(\sqrt{1-x^2})}$

$x = \cos t \Rightarrow \int_0^{\frac{\pi}{2}} \frac{dt}{1+y^2\cos^2 t} = \frac{1}{\sqrt{1+y^2}}\operatorname{arctg}\left(\frac{\operatorname{tg} t}{\sqrt{1+y^2}}\right)\Big|_0^{\frac{\pi}{2}} = \frac{\pi}{2\sqrt{1+y^2}} \to \int_0^1 \frac{\pi\,dy}{2\sqrt{1+y^2}} = \frac{\pi}{2}\ln\left|y+\sqrt{1+y^2}\right|\Big|_0^1$

$y = \operatorname{tg} u \qquad \ominus \int \sec(u)\,du$

$\Rightarrow dy = \sec^2 u\,du$

№ 3.2.4898

$$\int_0^1 \frac{\cos\frac{1}{1-x}}{\sqrt[n]{1-x^2}}dx \qquad \frac{1}{1-x}=t \to \frac{1}{t}=1-x,\ x=1-\frac{1}{t} \to dx=\frac{1}{t^2}dt$$

(4)

$$\Rightarrow \int_1^{\infty} \frac{\cos t}{\underbrace{\left(1-\left(1-\frac{1}{t}\right)^2\right)^{\frac{1}{n}}}_{\left(\frac{2}{t}-\frac{1}{t^2}\right)^{\frac{1}{n}}}}\cdot\frac{1}{t^2}dt$$

$$\int_0^1 \frac{\cos\frac{1}{1-x}}{\sqrt[n]{1-x}\sqrt[n]{1+x}}dx \qquad \int_0^1 \frac{\cos\frac{1}{1-x}}{\sqrt[n]{1-x}}dx=\int_1^{+\infty}\frac{\cos t}{t^{2-\frac{1}{n}}}dt \Rightarrow \text{из пр. Дирихле}$$

сх. $2-\frac{1}{n}>0$

Сведено о среднем + пр. Дирихле Абеля) т.е. $n<0,\ n>\frac{1}{2}$ сх.

Кт. работа

Р. сх. несобств. инт

Пример

$$F(y)=\int_0^{+\infty}\frac{\sin yx}{x}dx=\operatorname{sign} y\cdot\frac{\pi}{2},\ y\in\mathbb{R}$$

1) $y\in[a,b],\ a>0$, р. сх.

$\forall\varepsilon>0\ \exists\delta>0:\ \forall b>\delta:\ \left|\int_b^{+\infty} f(x,y)dx\right|<\varepsilon\ \ \forall y\in[a,b]$

$$\int_c^{+\infty}\frac{\sin xy}{x}dx=\left|\begin{matrix}xy=t\\ dx=\frac{dt}{y}\end{matrix}\right|=\left|\int_{cy}^{+\infty}\frac{\sin t}{t/y}\frac{dt}{y}\right|=\left|\int_{cy}^{+\infty}\frac{\sin t}{t}dt\right|\le\varepsilon$$

$cy\in[ca,cb]$, $\int_0^{+\infty}\frac{\sin t}{t}dt$ — сх. $\forall\varepsilon>0\ \exists\delta=\delta(\varepsilon)>0\ \forall c_1>\delta,\ \left|\int_{c_1}^{+\infty}\frac{\sin t}{t}dt\right|<\varepsilon$

$ca>c_1$

2) $y=0\in[a,b]$, нет р. сх.; отр. Коши; $c_1y=\frac{\pi}{4}$, $c_2y=\frac{\pi}{2}$; $c_1=2\pi n$, $y=\frac{1}{8n}$, $c_2=4\pi n$

$$\Rightarrow\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{\sin t}{t}dt\ge\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{1}{\sqrt{2}t}dt\ge A>0$$

$F(y)=\int_a^{+\infty}f(x,y)dx,\ y\in Y$

Пр-к Вейерштр.: Пусть 1) $|f(x,y)|\le g(x)$ $\forall y\in Y,\ x\in[a,+\infty)$

2) $g(x)\ge0$

3) $\int_a^{+\infty}g(x)dx$ сх. $\Rightarrow\int_a^{+\infty}f(x,y)$ сх. равн. на Y.

Пр-к Дирихле: $\int_a^{+\infty}f(x,y)g(x,y)dx$

1) $\left|\int_a^T f(x,y)dx\right|\le M\ \forall y\in Y\ \forall T\ge a$

2) $g(x,y)$ ~~↓~~ $\forall y\in Y$

3) $g(x,y)\underset{x\to\infty}{\rightrightarrows}0$

Пр-к Абеля: $\int_a^{+\infty}f(x,y)g(x,y)dx$

1) $\int_a^{+\infty}f(x,y)dx$ сх. р/м на Y

2) $|g(x,y)|\le C$ (не зависит от x,y)

3) ...

3) $g(x,y)$ монот. по x $\forall y \in Y$

Ф-ла ср. знач. $\int_A^B f_1(x) \cdot f_2(x)\,dx = f_2(A+0) \cdot \int_A^{\xi} f_1(x)\,dx + f_2(B-0) \cdot \int_{\xi}^{B} f_1(x)\,dx$, $f_2(x)$ монот. на $[a,b]$, $\xi \in [a,b]$.

№3656

$\int_0^{\infty} e^{-\alpha x} \sin x\,dx$ исп. на р.сх. $\alpha \in [0,+\infty)$

$\int_0^{+\infty} \frac{dx}{x^\alpha+1}$, $\int_a^{b} dx$ $1<\alpha<+\infty$; $\frac{1}{x^\alpha+1} \geq \frac{1}{2x^\alpha}$

$\int_b \geq \int \frac{1}{x^\alpha}\,dx = \frac{1}{2} \frac{x^{-(\alpha-1)}}{-\alpha+1} \Big|_b^{+\infty} = \frac{1}{2} \cdot \frac{b^{-\alpha+1}}{\alpha-1}$ $b=n$, $\alpha = 1+\frac{1}{n}$

$\frac{1}{2} \cdot n^{-\frac{1}{n}+1} \geq 1 = \varepsilon_0$

$n > n_0$

№3763

$\int_{-\infty}^{+\infty} e^{-(x-\alpha)^2}\,dx$ а) $a<\alpha<b$ б) $-\infty<a<+\infty$ на всей оси нет

б) $I(\alpha)$, $x-\alpha=t \Rightarrow \int_{-\infty}^{+\infty} e^{-t^2}\,dt = \sqrt{\pi}$, $\int_{-\infty}^{+\infty} e^{-x^2+2\alpha x-\alpha^2}\,dx \leq \int_{-\infty}^{+\infty} e^{-x^2+2bx}\,dx$ сх-ся.

р.сх. Вейерштр.

Др

№3754

$I = \int_0^{+\infty} \alpha e^{-\alpha x}\,dx$: 1) сх. р. на $\forall [a,b]$: $0<a\leq\alpha\leq b$

2) не р. сх. в $0 \leq \alpha \leq b$

$\int_{x_1}^{x_2} \alpha e^{-\alpha x}\,dx$ $\int_0^{+\infty} \alpha e^{-\alpha x}$ $|\alpha e^{-\alpha x}| \leq |\alpha| |e^{-ax}|$

2) $\forall \varepsilon>0\ \exists B:\ \left|\int_{b'}^{b''} f(x,y)\,dx\right| < \varepsilon$ $(y_1<y<y_2)$, $b'>B$, $b''>B$

$\exists \varepsilon>0\ \forall B:\ b'>B, b''>B:\ \left|\int_{b'}^{b''} f(x,y)\,dx\right| \geq \varepsilon$

$\exists \varepsilon>0\ \forall B\ \exists b \geq B\ \left|\int_b^{\infty} f(x,y)\,dx\right| \geq \varepsilon$ $(y_1<y<y_2)$ $\Rightarrow y_1, \exists y_2 > B$

2) $t=\alpha x$, $x>0 \wedge \alpha>0$

2) $\int_B^{\infty} \alpha e^{-\alpha x} = \int_{\alpha B}^{+\infty} e^{-t}\,dt = e^{-\alpha B}$

$\alpha \to 0 \Rightarrow e^{-\alpha B} \to 1$

$\Rightarrow \forall B>0\ \exists \alpha \in [0,b]: e^{-\alpha B} > \varepsilon$

$0<\varepsilon<1$

$\Rightarrow$ сх. не равн.

№3759

$\int_0^{+\infty} \frac{dx}{(x-\alpha)^2+1}$ $(0 \leq \alpha < +\infty)$ $\leq \int_0^1 + \int_1^{+\infty}$ по пр. Вейерштр. сх. равн.)

расх. сх.

$I(B,\alpha) = \int_{B-\alpha}^{+\infty} \frac{dt}{t^2+1}$ $\alpha = B \Rightarrow$ $\forall B$, $I(B,\alpha) > \varepsilon = \frac{\pi}{2} \Rightarrow$ не равн.-сх.

№ 3764

а) $\int_0^{+\infty} e^{-x^2(1+y^2)} \sin x \, dy$ $(-\infty < x < +\infty)$ ~~не пр. Дирихле р. сх.~~

$\frac{\sin x}{x} e^{-x^2} \cdot \frac{\sqrt{\pi}}{2}$, $A>0, x>0$

$\Rightarrow \int_A^{+\infty} e^{-x^2(1+y^2)} \sin x \, dy = \frac{\sin x}{x} e^{-x^2} \int_{Ax}^{+\infty} e^{-t^2} dt \xrightarrow[Ax\to 0]{} \int_0^{+\infty} e^{-t^2} dt \to \frac{\sqrt{\pi}}{2}$ $(x\to+0)$

б) $\int_0^{+\infty} \frac{\sin x^2}{1+x^p} dx$ $(p \geq 0)$ $\forall \varepsilon > 0 \ \exists b > 0 \ \forall \left| \int_b^{+\infty} \frac{\sin x^2}{1+x^p} dx \right| < \varepsilon \ \forall p \geq 0.$

$\int_0^{+\infty} \frac{1}{1+x^p} dx$; $\int_0^{+\infty} \sin x^2 dx$ | По пр. Абеля: $\int_0^{+\infty} = |x = \sqrt{t}| = \int_0^{+\infty} \frac{\sin t \, dt}{2(1+t^{\frac{p}{2}})\sqrt{t}}$

№ 3766

$\int_0^1 x^{p-1} \ln^q \frac{1}{x} dx$ а) $p \geq p_0 > 0$; б) $p > 0$ $(q > -1)$, $\int_0^1 = |e^{-t} = x| = \int_0^{+\infty} \cancel{x^{p-1}} e^{-(p-1)t} t^q e^{-t} dt =$

$dt = \ln x$, $t = -\ln x$

$= \int_0^{+\infty} e^{-pt} t^q dt$

а) $t^q e^{-pt} \leq t^q e^{-p_0 t}$ $\int_0^{+\infty} t^q e^{-p_0 t} dt$ сх. по пр. ср. $\Rightarrow$ пр. Вейерштрасса сх. равн.

б) $I(b,p) = \int_b^{+\infty} t^q e^{-pt} dt$, $b > 0$ $z = pt$ $\Rightarrow I(b,p) = \frac{1}{p^{q+1}} \int_{bp}^{+\infty} z^q e^{-z} dz$

Пусть $b > 0$, $\varepsilon > 0$ задан. $\Rightarrow \lim_{p\to+0} I(b,p) = +\infty$

$\Rightarrow \exists p > 0 : I(b,p) > \varepsilon$.

№ 3768

$\int_0^1 \sin \frac{1}{x} \frac{dx}{x^n}$ $(0 < n < 2)$, $x = \frac{1}{t}$ $\Rightarrow \int_0^1 \sin \frac{1}{x} \frac{dx}{x^n} = \int_1^{+\infty} \sin t \cdot t^{n-2} dt =$

$= \int_b^{+\infty} \frac{\sin t}{t^{2-n}} dt = \frac{\cos b}{b^{2-n}} + (n-2) \int_b^{+\infty} \frac{\cos t}{t^{3-n}} dt$

$\left| \frac{\cos b}{b^{2-n}} \right| = \frac{1}{(2\pi k)^{2-n}} > \varepsilon_2$ $\quad$ $\frac{1}{t} \to 0$ $|S| \leq C$ | $\Rightarrow$ сх. равн. $\Rightarrow \exists b : \left| \int_b \right| < \varepsilon$,

$b = 2\pi k$, любые n | $\Rightarrow$ не равн.

Кр. работа

$F(y)=\int\limits_a^{+\infty} f(x,y)dx,\ y\in Y$

1) $f(x,y)\in C[a\le x<+\infty, y\in Y]$
2) $\int\limits_a^{+\infty} f(x,y)dx$ сх. р. на Y
$\to F(y)\in C(Y)$

---

1) $\exists$ т.: $\int\limits_a^{+\infty} f(x,y)dx$ сх. в т. $y_0\in[a,b]$
2) $f(x,y)\in C[a\le x<+\infty, y\in[a,b]]$
3) $\exists f'_y(x,y)\in C[y\in[a,b]]$
4) $\int\limits_a^{+\infty} f'_y(x,y)dx$ сх. равн. на $[a,b]$

$\Rightarrow \exists F'(y)=\int\limits_a^{+\infty} f'_y(x,y)dx$

№ 3779

$\int\limits_0^{+\infty}\frac{x\,dx}{2+x^\alpha}=F(\alpha),\ \alpha>2$

$\alpha_0>2:\ \exists\alpha_1:\alpha_1>2,\ \alpha_1<\alpha_0 \Rightarrow$ есть р. сх. на мн-ве $[\alpha_1,+\infty)$

$F(\alpha)=\underbrace{\int\limits_0^1\frac{x\,dx}{2+x^\alpha}}_{\in C(2,+\infty)}+\int\limits_1^{+\infty}\frac{x\,dx}{2+x^\alpha},\quad x\ge1:\ \frac{x}{2+x^\alpha}\le\frac{x}{2+x^{\alpha_1}}\quad \alpha\ge\alpha_1 \Rightarrow$ по пр. Вейерштр. сх. равном. $\int\limits_1^{+\infty}\frac{x\,dx}{2+x^\alpha}$ на $[\alpha_1,+\infty)$

$\Rightarrow F(\alpha)\in C[\alpha_1,+\infty)\to F(\alpha)\in C(\alpha_0)\to F(\alpha)\in C(2,+\infty)$

сх. равн. на $(2,+\infty)$: отр. $\to \left|\int\limits_b^{+\infty}\frac{x\,dx}{2+x^\alpha}\right|\ge\varepsilon$,

р/н? $b>2$, $\int\limits_b^{+\infty}\frac{x\,dx}{2+x^\alpha}\ge\int\limits_b^{+\infty}\frac{x\,dx}{2\cdot x^\alpha}=\frac{1}{2}\left(\frac{x^{2-\alpha}}{2-\alpha}\right)\Big|_{x=b}^{x=+\infty}=\frac{1}{2}\cdot\frac{b^{2-\alpha}}{\alpha-2}$, $\alpha=2+\frac{1}{n}$

$b=n$

$\frac{1}{2}\cdot\frac{n^{-\frac{1}{n}}}{\frac{1}{n}}=\frac{1}{2}n^{1+\frac{1}{n}}\ge1$

$\varepsilon=1$

$\forall n\ge n_0$

№ 3782

$F(\alpha)=\int\limits_0^{+\infty}\frac{e^{-x}}{|\sin x|^\alpha}dx,\ 0<\alpha<1,\quad F(\alpha)=\sum\limits_{k=0}^{\infty}\int\limits_{\pi k}^{\pi(k+1)}\frac{e^{-x}}{|\sin x|^\alpha}dx\ (**)$

$\int\limits_{\pi k}^{\pi(k+1)}\frac{e^{-x}}{|\sin x|^\alpha}dx=\left|\begin{matrix}x=\pi k+t\\ dx=dt\end{matrix}\right|=\int\limits_0^{\pi}\frac{e^{-\pi k+t}}{(\sin t)^\alpha}dt=e^{-\pi k}\underbrace{\int\limits_0^{\pi}\frac{e^{-t}}{(\sin t)^\alpha}dt}_{(*)}$

$(*)$ сх. $\Rightarrow (**)=\sum\limits_{k=0}^{\infty}e^{-\pi k}\int\limits_0^{\pi}\frac{e^{-t}}{(\sin t)^\alpha}dt=\int\limits_0^{\pi}\frac{e^{-t}}{(\sin t)^\alpha}dt\left[\sum\limits_{k=0}^{\infty}e^{-\pi k}=C\right]\int\limits_0^{\pi}\frac{e^{-t}}{(\sin t)^\alpha}dt=$

$=C\left(\int\limits_0^{\pi/2}\_\_+\int\limits_{\pi/2}^{\pi}\_\_\right)$ $\forall\varepsilon>0\ \exists\delta(\varepsilon)>0\ \forall b: b<\delta,\ \forall\alpha: 0<\alpha<1$

$\int\limits_0^b\frac{e^{-t}}{(\sin t)^\alpha}dt<\varepsilon$ $\quad \exists\varepsilon>0\ \forall\delta>0\ \exists\ 1<b<\delta\ \exists\alpha: 0<\alpha<1\ \int\limits_0^b\_\_\ge\varepsilon$

$\delta<1: \int_0^{\delta} \frac{e^{-t}}{(\sin t)^\alpha}dt > e^{-\delta} C \int_0^{\delta} \frac{1}{t^\alpha}dt = e^{-\delta} C \frac{\delta^{1-\alpha}}{1-\alpha}$ $\quad \alpha=1$ $\to e^{-\frac{1}{n}} C \left(\frac{1}{n}\right)^{\frac{1}{n}}$ $n \ge 1$

$\frac{2}{\pi} t \le \sin t \le t$ $\quad 1-\frac{1}{n}=\alpha$, $\delta=\frac{1}{n}$ $\quad \forall n \ge n_0$

$\int_0^{\delta} \frac{e^{-t}}{(\sin t)^\alpha} \ge \varepsilon$ на $(0;\alpha_1]$ если р/м сх.:

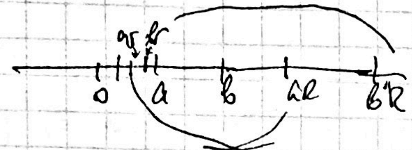
№ 3789

Формула Фруллани!

$\int_0^{+\infty} \frac{f(ax)-f(bx)}{x}dx = f(0)\cdot \ln\frac{b}{a}$, $a,b>0$ | если $f(x)\in C[0,+\infty)$, $\exists \int_A^{+\infty}\frac{f(x)dx}{x}$ $\forall A>0$

$I(a,b)$ $\quad I(a,b) = \lim_{\substack{r\to+0 \\ R\to+\infty}} \int_r^R \frac{f(ax)-f(bx)}{x}dx$

$\int_r^R \frac{f(ax)-f(bx)}{x}dx = \int_r^R \frac{f(ax)}{x}dx - \int_r^R \frac{f(bx)}{x}dx = \int_{ar}^{aR}\frac{f(t)}{t}dt - \int_{br}^{bR}\frac{f(t)dt}{t} \ominus$



$\ominus \int_{ar}^{br}\frac{f(t)}{t}dt - \int_{aR}^{bR}\frac{f(t)dt}{t}$; $\lim_{\substack{r\to0 \\ R\to+\infty}} \int_{aR}^{bR}\frac{f(t)dt}{t} = 0$ (из сх. интеграла)

$\lim_{r\to 0}\int_{ar}^{br}\frac{f(t)dt}{t} \ominus$

$\ominus \lim_{r\to+0} f(\xi)\cdot\int_{ar}^{br}\frac{dt}{t} = \lim_{r\to+0} f(\xi)\ln\frac{b}{a} = f(0)\ln\frac{b}{a}$, $\xi\in[ar, br]$

№ 3792

$\int^{+\infty}\frac{\operatorname{arctg}x}{x}dx$ не $\exists$ $\quad \int_0^{+\infty}$ $f(x) = \operatorname{arctg}x - \frac{\pi}{2}$

$\int_A^{+\infty}\frac{\operatorname{arctg}x-\frac{\pi}{2}}{x}dx$, $\operatorname{arctg}x-\frac{\pi}{2} \sim -\frac{1}{x}$, $x\to\infty$ | $\Rightarrow$ сход. Фрул.

$-\operatorname{arcctg}x \to x\to 0$

$I(a,b) = -\frac{\pi}{2}\ln\frac{b}{a} = \frac{\pi}{2}\ln\frac{a}{b}$

№ 3776(2)

$f(x)$ — непр., огр. на $[0,+\infty)$ $\lim_{y\to+0}\frac{2}{\pi}\int_0^{+\infty}\frac{yf(x)}{x^2+y^2}dx = f(0)$

$\frac{1}{y}\int_0^{\infty}\frac{f(x)dx}{1+\left(\frac{x^2}{y^2}\right)} = \frac{1}{y}\int_0^{\infty}\frac{f(y\cdot t)\cdot y\,dt}{1+t^2} = \int_0^{+\infty} f(y\cdot t)\frac{1}{1+t^2}dt \le f(yt_0)\cdot \operatorname{arctg}t\Big|_0^{\infty}$ $\left(=\frac{\pi}{2}\right)$

$\left| \begin{array}{l} x = y\cdot t \\ dx = y\cdot dt \end{array} \right|$ $\Rightarrow$ сх. $\Rightarrow$ непр. $\frac{f(ty)}{t^2+1}$ $\Rightarrow \lim_{y\to+0}\frac{2}{\pi}\int_0^{+\infty}\frac{f(ty)}{t^2+1}dt =$

$= \frac{2}{\pi}\int_0^{+\infty}\lim_{y\to+0}\frac{f(ty)}{t^2+1}dt = f(0).$

~~Сх/Р~~

№ 3776(2)

№3780

$F(\alpha)=\int_1^{+\infty}\frac{\sin x}{x^\alpha}dx,\ \alpha>0$ непр.?

По пр. Дирихле сх. р.⇒ непр.

№3781

$F(\alpha)=\int_0^{\pi}\frac{\sin x}{x^\alpha(\pi-x)^\alpha}dx,\ 0<\alpha<2$

$F(\alpha)=\int_0^{\frac{\pi}{2}}\frac{\sin x}{x^\alpha(\pi-x)^\alpha}+\int_{\frac{\pi}{2}}^{\pi}\frac{\sin x}{x^\alpha(\pi-x)^\alpha}=\int_0^{\frac{\pi}{2}}\ldots-\int_{\frac{\pi}{2}}^{0}\frac{\sin(\pi-t)}{(\pi-t)^\alpha t^\alpha}dt=2\int_0^{\frac{\pi}{2}}\frac{\sin x}{x^\alpha(\pi-x)^\alpha}dx$

$\exists\ 0<\varepsilon\le\alpha\le 2-\varepsilon<2$

$I\le\int_0^1\frac{dx}{x^{\alpha-1}(\pi-x)^\alpha}+\int_1^{\pi-1}\frac{dx}{x^\alpha(\pi-x)^\alpha}+\int_{\pi-1}^{\pi}\frac{dx}{x^\alpha(\pi-x)^{\alpha-1}}\le\int_0^1\frac{dx}{x^{1-\varepsilon}}+\pi-2+\int_{\pi-1}^{\pi}\frac{dx}{(\pi-x)^{1-\varepsilon}}$

⇒ по пр. ср. они сх. ⇒ по пр. Вейерштрасса р. сх. $\varepsilon<\alpha\le 2-\varepsilon$

и т.к. $f(x,y)$ непр. ⇒ $F(\alpha)$ непр.

№3783

$F(\alpha)=\int_0^{+\infty}\alpha e^{-x\alpha^2}dx\quad -\infty<\alpha<+\infty;\quad -b\le\alpha\le b$

$|\alpha e^{-x\alpha^2}|\le x e^{-x\alpha^2};\quad F(\alpha)\underset{\alpha\ne0}{=}-\frac{1}{\alpha}e^{-x\alpha^2}\Big|_0^{+\infty}=\frac{1}{\alpha},\quad \alpha=0\Rightarrow F(0)=0$

⇒ не явл. непр.

№3787

$F(\alpha)=\int_0^{+\infty}\frac{\cos x}{1+(x+\alpha)^2}dx$ непр. и диф. в обл. $-\infty<\alpha<+\infty$

$\left(\frac{\cos x}{(1+(x+\alpha)^2)}\right)'_\alpha=-\cos x\frac{1}{(1+(x+\alpha)^2)^2}2(x+\alpha)$ р. непр. ⇒ д. д.

№3793

$F(\alpha,\beta)=\int_0^{+\infty}\frac{e^{-\alpha x^2}-e^{-\beta x^2}}{x}dx\quad(\alpha>0,\beta>0)\qquad F'_\alpha=\int_0^{+\infty}\left(-\cancel{\frac{e^{-\beta x^2}}{x}}+\cancel{\frac{1}{x}e^{-\alpha x^2}}\right)dx=$

$=\int_0^{+\infty}-xe^{-\alpha x^2}dx=\frac{1}{2\alpha}\int_0^{+\infty}e^{-\alpha x^2}d(\ldots)$

$\cancel{=\frac{1}{2\alpha}\frac{\sqrt{\pi}}{2}=\frac{\sqrt{\pi}}{4\alpha^2}}$

⇒ $F(\alpha,\beta)=\frac{\sqrt{\pi}}{4}\ln\alpha+f(\beta)$

~~$F(1,\beta)=f(\beta)=\int_0^{+\infty}$~~ $F(\beta,\beta)=0=\frac{\sqrt{\pi}}{4}\ln\beta+f(\beta)\Rightarrow f(\beta)=-\frac{\sqrt{\pi}}{4}\ln\beta$

⇒ $F(\alpha,\beta)=\frac{\sqrt{\pi}}{4}\ln\frac{\alpha}{\beta}$

Кр. работа

1) $\int_0^{+\infty} e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$ – інтегр. Пуассона 4) $\int_0^{+\infty} e^{-(x^2+\frac{a^2}{x^2})}dx = \frac{\sqrt{\pi}}{2}e^{-2|a|}$

2) $\int_0^{+\infty}\frac{\sin \alpha x}{x}dx = \operatorname{sgn}\alpha\cdot\frac{\pi}{2}$ – интегр. Дирихле 5) $\int_0^{+\infty}\sin(x^2)dx = \int_0^{+\infty}\cos(x^2)dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$ – инт. Френеля

3) $\int_0^{+\infty} e^{-x^2}\cos(bx)dx = \frac{1}{2}\sqrt{\pi}e^{-b^2/4}$ 6) $\int_0^{+\infty}\frac{\cos \alpha x}{a^2+x^2}dx \ominus a>0, \alpha\in\mathbb{R} \ominus \frac{\pi}{2a}e^{-a|\alpha|}$

$-I'(b) = +\int_0^{+\infty} e^{-x^2}\sin(bx)x\,dx = -\frac{e^{-x^2}}{2}\sin bx\Big|_0^{+\infty} + \int_0^{+\infty}\frac{e^{-x^2}}{2}\cos(bx)b\,dx = \frac{b}{2}\int_0^{+\infty}e^{-x^2}\cos(bx)dx = \frac{b}{2}I(b)$

$I(b) = -\frac{2}{b}I'(b)$ $b\,db = -2\frac{dI}{I}$ $\Rightarrow I(b) = Ce^{-b^2/4}$, $C = \frac{\sqrt{\pi}}{2} \Rightarrow I(b) = \frac{\sqrt{\pi}}{2}e^{-\frac{b^2}{4}}$.

$I(a) = \int_0^{+\infty}e^{-(x^2+\frac{a^2}{x^2})}dx$, $a>0$, $I(a) = \int_0^{+\infty}e^{-(\frac{1}{t^2}+a^2t^2)}\cdot\frac{1}{t^2}dt = \left|\begin{array}{l} at=y \\ t=\frac{y}{a} \\ dt=\frac{dy}{a}\end{array}\right| = \int_0^{+\infty}e^{-(y^2+\frac{a^2}{y^2})}\frac{a}{y^2}dy$

$\frac{1}{x} = t$, $dx = -\frac{1}{t^2}dt$

$I'(a) = -\int_0^{+\infty}e^{-(x^2+\frac{a^2}{x^2})}\frac{2a}{x^2}dx = I'(a)$, $a\in[\varepsilon, A]$, $\varepsilon>0$, $\forall A\in\mathbb{R}$; $I'(a) = -2 I(a)$

$\frac{dI}{da} = -2I$;

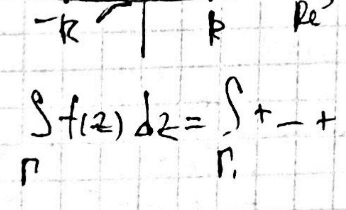
$\frac{dI}{I} = -2da \Rightarrow I(a) = \frac{\sqrt{\pi}}{2}e^{-2|a|}$

$\int_0^{+\infty}\sin(x^2)dx = \left|\begin{array}{l} t=x^2 \\ x=\sqrt{t} \\ dx=\frac{1}{2\sqrt{t}}dt\end{array}\right| = \int_0^{+\infty}\sin t\frac{1}{2\sqrt{t}}dt =$

$I = \int_0^{+\infty}\frac{\sin(x^2)}{2\sqrt{t}}dx = \frac{2}{\sqrt{\pi}}\int_0^{+\infty}\sin t\left(\int_0^{+\infty}e^{-tu^2}du\right)dt \overset{?}{=} \frac{2}{\sqrt{\pi}}\int_0^{+\infty}du\int_0^{+\infty}\sin t\,e^{-tu^2}dt \xrightarrow[a=u^2]{b=1} \frac{2}{\sqrt{\pi}}\int_0^{+\infty}du\frac{1}{u^4+1}$

$\int_0^{+\infty}e^{-ax}\sin(bx)dx = \frac{b}{a^2+b^2}$ (№2349), $a>0$

$\int_0^{+\infty}\frac{du}{u^4+1} = \frac{1}{2}\int_{-\infty}^{+\infty}\frac{du}{u^4+1} = \frac{1}{2}\lim_{R\to\infty}\int_{-R}^{R}\frac{du}{u^4+1} =$



$f(z) = \frac{1}{z^4+1}$, $z^4+1=0$, $z^4=-1 \Leftrightarrow z^2 = i$, $\Leftrightarrow z = e^{\frac{i\pi+2\pi k i}{4}}$, $z_1 = e^{\frac{i\pi}{4}}$, $z_2 = e^{\frac{3\pi i}{4}}$

$z_3 =$ —

$z_4 =$ —

$\int_\Gamma f(z)dz = \int_{\Gamma_1} + \int_{\Gamma_2} = 2\pi i\left(\frac{1}{(z_1-z_2)(z_1-z_3)(z_1-z_4)} + \frac{1}{(z_2-z_1)(z_2-z_3)(z_2-z_4)}\right) \textcircled{=}$

$f(z) = z^4+1 = (z-z_1)(z-z_2)(z-z_3)(z-z_4)$, $z\neq z_1$: $\frac{f(z)}{z-z_1} = (z-z_2)(z-z_3)(z-z_4)$

$\Rightarrow \lim_{z\to z_1}\frac{f(z)}{z-z_1} = (z_1-z_2)(z_1-z_3)(z_1-z_4) = f'(z_1) = 4z_1^3$

$\textcircled{=} 2\pi i\left(\frac{1}{4z_1^3}+\frac{1}{4z_2^3}\right) = \frac{\pi i}{2}\left(e^{-\frac{3\pi i}{4}}+e^{-\frac{9\pi i}{4}}\right) = \frac{\pi i}{2}\left(-2\cdot\frac{1}{\sqrt{2}}i\right) = \frac{\pi}{\sqrt{2}}$

$\Rightarrow \frac{2}{\sqrt{\pi}}\cdot\frac{\pi}{2\sqrt{2}} = \sqrt{\frac{\pi}{2}} = I$

(?) : Вообще говоря нельзя

$F(y) = \int_0^{+\infty} f(x,y)dx$, $y \in [c,d]$, $\exists$ 1) $f(x,y) \in C([a,+\infty) \times [c,d])$
2) $\int^{+\infty}$ сх. р. на $\forall$.

$\Downarrow$

$$\int_c^d dy\, F(y) = \int_a^{+\infty} dx \int_c^d f(x,y)dy$$

$F(y) = \_\_\_$, $y \in [c,+\infty)$ $\exists$ 1) $f(x,y) \geq 0$ на $[a;+\infty) \times [c;+\infty)$
2) $F(y)$ - непр. ф. на $[c,+\infty)$
3) $\int_c^{+\infty} f(x,y)dy$ - непр. ф. на $[a,+\infty)$
4) $\exists \int_c^{+\infty} dy [\int_a^{+\infty} f(x,y)dx]$

$\exists \int_0^{+\infty} dx [\int_c^{+\infty} f(x,y)dy]$ и они $=$

1) $\exists f(x,y) \in C[a;+\infty) \times [c;+\infty)$
2) $\int^{+\infty} f(x,y)dx$ - сх-ся равн. на $\forall$ огр. $[c,d]$.
3) $\int_a^{+\infty} f(x,y)\, dy$ - сх-ся равн. на $\forall$ огр. $[a,b]$
4) $\exists \int_a^{+\infty} dx [\int_c^{+\infty} |f(x,y)|dy]$ или $\exists \int_c^{+\infty} dy \int_a^{+\infty} |f(x,y)| dx$

$\Downarrow$

$\exists$ и 2-й инт. и они равны.

Переходим в пределе $[\varepsilon, \_\_\_)$

$\int_0^{+\infty} \frac{\sin t}{\sqrt{t}} e^{-kt} dt$, $k \geq 0$. сх. равн. по k. ; $\int_0^{+\infty} \frac{\sin t}{\sqrt{t}} dt = \lim_{k \to 0+} \int_0^{+\infty} \frac{\sin t}{\sqrt{t}} e^{-kt} dt$

$\int_0^{+\infty} \frac{\sin t}{\sqrt{t}} e^{-kt} dt = \frac{2}{\sqrt{\pi}} \int_0^{+\infty} e^{-kt} \sin t\, dt \left[\int_0^{+\infty} e^{-tu^2} du\right] = \frac{2}{\sqrt{\pi}} \int_0^{+\infty} du \int_0^{+\infty} \sin t\, e^{-t(u^2+k)} dt =$

$= \frac{2}{\sqrt{\pi}} \int_0^{+\infty} \frac{du}{1+(k+u^2)^2} \xrightarrow[k \to 0+]{} \frac{2}{\sqrt{\pi}} \int_0^{+\infty} \frac{du}{1+u^4}$

$L(a,\alpha) = \int_0^{+\infty} \frac{\cos(\alpha x)}{a^2+x^2} dx = \int_0^{+\infty} dx \left(\int_0^{+\infty} e^{-(a^2+x^2)y} \cos(\alpha x)\, dy\right) = \int_0^{+\infty} dy \left(\int_0^{+\infty} e^{-(a^2+x^2)y} \cos(\alpha x)\, dx\right) =$

$= \int_0^{+\infty} e^{-a^2 y} dy \left(\int_0^{+\infty} e^{-x^2 y} \cos(\alpha x)\, dx\right) = \int_0^{+\infty} e^{-a^2 y} dy \left(\int_0^{+\infty} e^{-(x\sqrt{y})^2} \cos\left(\frac{\alpha}{\sqrt{y}}(x\sqrt{y})\right) dx\right) = \left| \begin{array}{l} t = x\sqrt{y} \\ x = t/\sqrt{y} \\ dx = \frac{dt}{\sqrt{y}} \end{array} \right| =$

$$\ominus \int_0^{+\infty} \frac{e^{-a^2 y}}{\sqrt{y}} \cdot \frac{1}{2}\sqrt{\pi}\, e^{-\frac{\alpha^2}{4y}} dy = \frac{\sqrt{\pi}}{2}\int_0^{+\infty} e^{-(a^2 y + \frac{\alpha^2}{4y})}\cdot \frac{1}{\sqrt y}\, dy = \left| \begin{array}{l} z=\sqrt y \\ y=z^2 \\ dy = 2z\,dz \end{array}\right| = \sqrt{\pi}\int_0^{+\infty} e^{-(a^2z^2+\frac{\alpha^2}{4z^2})}\, dz \ominus$$

$$\ominus \left| \begin{array}{l} az=u \\ dz=\frac{du}{a} \end{array}\right| = \sqrt\pi \int_0^{+\infty} e^{-(u^2+\frac{\alpha^2 a^2}{4}\cdot\frac{1}{u^2})}\,\frac{du}{a} = \frac{\sqrt\pi}{a}\cdot\frac{\sqrt\pi}{2} e^{-2\left|\frac{\alpha a}{2}\right|} = \frac{\pi}{2a} e^{-a|\alpha|}$$

Д/р

№ 3805

$$\int_{-\infty}^{+\infty}(a_1x^2+2b_1x+c_1)e^{-(ax^2+2bx+c)}dx \quad (a>0,\ ac-b^2>0)$$

$\frac{1}{\sqrt a}(ax+b)=t \quad x=\frac{\sqrt a\, t - b}{a}$, $\displaystyle\int_{-\infty}^{+\infty} \_\_ = \frac{1}{\sqrt a} e^{\frac{b^2-ac}{a}} \int_{-\infty}^{+\infty}\left[\frac{a_1}{a}t^2 + \frac{2(ab_1-a_1b)}{a\sqrt a}\,t +\right.$

$\left.+ \frac{a_1b^2-2abb_1}{a^2}+c_1\right]e^{-t^2}dt$; $\displaystyle\int_{-\infty}^{+\infty} t^2 e^{-t^2} = \frac{\sqrt\pi}{2}$ по частям

$\displaystyle\int_{-\infty}^{+\infty} t e^{-t^2}dt=0$;

$\displaystyle\int_{-\infty}^{+\infty} e^{-t^2}dt=\sqrt\pi \quad \Rightarrow$ ___

№ 3808

$$\int_0^{+\infty}\frac{e^{-\alpha x^2}-e^{-\beta x^2}}{x^2}dx\quad (\alpha>0,\ \beta>0) = -\int_0^{+\infty}(e^{-\alpha x^2}-e^{-\beta x^2})\,d\left(\frac1x\right) \ominus$$

$$\ominus -\frac{e^{-\alpha x^2}-e^{-\beta x^2}}{x}\Big|_0^{+\infty} - 2\int_0^{+\infty}(\alpha e^{-\alpha x^2}-\beta e^{-\beta x^2})dx = -2\sqrt\alpha\,\frac{\sqrt\pi}{2}+2\sqrt\beta\,\frac{\sqrt\pi}{2}$$

№ 3816

$\displaystyle\int_0^{+\infty}\frac{\sin^3\alpha x}{x}dx$ $\quad\Big|\quad$ $\displaystyle\int_0^{+\infty}\frac{f(ax)-f(bx)}{x} = [f(0)-\cancel{f(\infty)}]\ln\left(\frac ba\right)$

$\displaystyle\int_0^{+\infty}\frac{\sin x}{x}dx = \frac\pi2$

$$\sin 3\alpha x = 3\sin\alpha x - 4\sin^3\alpha x \Rightarrow \int \_\_ = \int_0^{+\infty}\frac{3\sin\alpha x-\sin3\alpha x}{4x}dx =$$

$$=\frac14\Bigg(\underbrace{\int_0^{+\infty}\frac{\sin\alpha x-\sin 3\alpha x}{x}}_{(0)} + 2\underbrace{\int_0^{+\infty}\frac{\sin\alpha x}{\alpha x}\,d(\alpha x)}_{\pm\frac\pi2}\Bigg) = \frac\pi4 \operatorname{sgn}\alpha$$

№ 3827

$$\int_0^{+\infty}\frac{\sin^2 x}{1+x^2}dx = \int\frac{1-\cos2x}{2(1+x^2)} = \frac12\left(\int\frac{1}{1+x^2} - \_\_\right) = \frac\pi4(1-e^{-2})$$

√ ~~3808~~

$$\int_0^{+\infty}\frac{\cos\alpha x}{(1+x^2)^2}dx = \int_0^{+\infty}\frac{\cos\alpha x}{1+x^2}dx - \int_0^{+\infty}\frac{x^2\cos\alpha x}{(1+x^2)^2}dx = \frac{\pi}{2}e^{-|\alpha|} + \frac{1}{2}\int_0^{+\infty}x\cos\alpha x\, d\left(\frac{1}{1+x^2}\right)$$

$$= \frac{\pi}{2}e^{-|\alpha|} + \frac{1}{2}\frac{x\cos\alpha x}{1+x^2}\Big|_0^{+\infty} - \frac{1}{2}\int_0^{+\infty}\frac{\cos\alpha x - \alpha x\sin\alpha x}{1+x^2}dx = \frac{\pi}{2}e^{-|\alpha|} - \frac{1}{2}\int_0^{+\infty}\frac{\cos\alpha x}{1+x^2}dx +$$

$$+\frac{\alpha}{2}\underbrace{\int_0^{+\infty}\frac{x\sin\alpha x}{1+x^2}dx}_{I_1} = \frac{\pi}{2}e^{-|\alpha|} - \frac{\pi}{4}e^{-|\alpha|} + \frac{\alpha}{2}\cdot\frac{\pi}{2}\operatorname{sgn}\alpha\cdot e^{-|\alpha|} = \frac{\pi}{4}(1+|\alpha|)e^{-|\alpha|}$$

I_1 $\quad \frac{\partial}{\partial\alpha}\left(\frac{\cos\alpha x}{1+x^2}\right) = -\frac{x\sin\alpha x}{(1+x^2)}$; $\Rightarrow I_1 = -\frac{dI}{da} = \frac{\pi}{2}e^{-a}\ (a>0)$

$I_1 = -\frac{\pi}{2}e^{a},\ a<0$

$a=0,\ I_1=0 \Rightarrow I_1 = \frac{\pi}{2}\operatorname{sgn}\alpha\, e^{-|a|}$.

√ 3811

$$\lim_{x\to+\infty}\sqrt{x}\int_{-\delta}^{\delta}e^{-axt^2}dt = \sqrt{\frac{\pi}{a}}\quad(a>0,\ \delta>0).$$

$1 - axt^2 + \frac{1}{2}a^2x^2t^4 + \dots$

$\sqrt{x}\left(\cancel{t} - 2ax\frac{t^3}{3} + \frac{a^2x^2}{4}t^5 + \dots\right)$

Кл. работа

√ 3841

$$\int_0^1\sqrt{x-x^2}\,dx = \int_0^1 x^{\frac12}(1-x)^{\frac12}dx = B\left(\frac32,\frac32\right) = \frac{\Gamma^2\left(\frac32\right)}{\Gamma(3)} = \frac{\left(\frac12\Gamma\left(\frac12\right)\right)^2}{\cancel{\Gamma\left(2+\frac12\right)}} \frac{\frac14\pi}{2} =$$

$= \frac{\pi}{8}.$

√ 3879

$$\int_0^{\frac{\pi}{2}}\sin^6x\cos^4x\,dx = \int_0^{\frac{\pi}{2}}(\sin^2x)^3(1-\sin^2x)^2dx = \left|\begin{array}{l} t=\sin x\\ x=\arcsin t\\ dx=\frac{1}{\sqrt{1-t^2}}dt\end{array}\right| = \int_0^1 (t^2)^3(1-t^2)^2\cdot(1-t^2)^{-\frac12}dt =$$

$$= \int_0^1(t^2)^3(1-t^2)^{\frac32}dt = \left|\begin{array}{l} t^2=u\\ t=\sqrt{u}\\ dt=\frac{1}{2\sqrt{u}}du\end{array}\right| =$$

$$= \int_0^1 u^3(1-u)^{\frac32}\cdot 2u^{-\frac12}du = 2\int_0^1 u^{\frac52}(1-u)^{\frac32}du = 2B\left(\frac72,\frac52\right) = 2\frac{\overbrace{\Gamma\left(\frac72\right)}^{\Gamma(3+\frac12)}\cdot\overbrace{\Gamma\left(\frac52\right)}^{\Gamma(2+\frac12)}}{\Gamma(6)\ 6!} =$$

$$= 2\frac{\frac{3\cdot5}{2^3}\cdot\frac{3}{2^2}\pi}{2\cdot3\cdot4\cdot5\cdot6} = \frac{\cancel{3}\cdot\cancel{5}\cdot\cancel{3}\,\pi}{2^5\cdot\cancel{3}\cdot4\cdot\cancel{5}\cdot\cancel{6}_2} = \frac{1}{2^5\cdot4\cdot2}\pi = \frac{1}{2^8}\ \frac{3\pi}{512}$$

№ 3868

$$\int_0^1 \ln\Gamma(x)\,dx = x\ln\Gamma(x)\Big|_0^1 - \int_0^1 x\cdot\frac{1}{\Gamma(x)}\Gamma'(x)\,dx$$

№ 3870

$$\int_0^1 \ln\Gamma(x)\sin\pi x\,dx = |x=1-t| \Rightarrow I = \frac{1}{2}\int_0^1 \ln(\Gamma(x)\Gamma(1-x))\sin\pi x\,dx$$

№ 3850

$$\int_0^{+\infty} x^{2n}e^{-x^2}dx \quad n - \text{цел. пол.} = \left| \begin{array}{l} x^2=t \\ x=\sqrt{t} \\ dx=\frac{1}{2}\frac{1}{\sqrt{t}}dt \end{array} \right| = \int_0^{+\infty} t^n e^{-t}\frac{1}{2}\cdot t^{-\frac{1}{2}}dt = \frac{1}{2}\int_0^{+\infty} t^{n-\frac{1}{2}}e^{-t}dt$$

$$= \frac{1}{2}\Gamma\left(n+\frac{1}{2}\right)$$

$$= \frac{(2n-1)!!}{2^{n+1}}\sqrt{\pi}.$$

№ 3851

$$\int_0^{+\infty} \frac{x^{m-1}}{1+x^n}dx \quad (n>0) = \left| \begin{array}{l} x^n=t \\ x=t^{\frac{1}{n}} \\ dx=\frac{1}{n}t^{\frac{1}{n}-1} \end{array} \right| = \frac{1}{n}\int_0^{+\infty}\frac{t^{\frac{m}{n}-1}}{1+t}dt =$$

$$B(p,q) = \int_0^{+\infty}\frac{x^{p-1}}{(1+x)^{p+q}}dx$$

$$= \frac{1}{n}B\left(\frac{m}{n}, 1-\frac{m}{n}\right) = \frac{1}{n}\Gamma\left(1-\frac{m}{n}\right)\Gamma\left(\frac{m}{n}\right) =$$

$$= \frac{\pi}{n\sin\frac{\pi m}{n}}.$$

№ 3861

$$\int_0^1 \underbrace{\left(\ln\frac{1}{x}\right)}_{=t}{}^p dx = \int_0^{+\infty} t^p e^{-t}dt = \Gamma(p+1), \quad p>-1.$$

Д/з

№ 3844

$$\int_0^a x^2\sqrt{a^2-x^2}\,dx \quad (a>0) = |x=a\sqrt{t}| = \frac{a^4}{2}\int_0^1 t^{\frac{1}{2}}(1-t)^{\frac{1}{2}}dt = \frac{a^4}{2}B\left(\frac{3}{2};\frac{3}{2}\right) =$$

$= \frac{\pi^4}{16}$.

3845

$\int_0^{+\infty} \frac{x^{\frac{1}{4}}}{(1+x)^2} dx$, $B(p,q) = \int_0^{+\infty} \frac{x^{p-1}}{(1+x)^{p+q}} dx$ ——

$= B$

3847

$\int_0^{+\infty} \frac{x^2 dx}{1+x^4} =$

3849

$\int_0^1 \frac{dx}{\sqrt[n]{1-x^n}}$ $(n>1)$ $= \left| \begin{array}{l} x^n = t \\ x = t^{\frac{1}{n}} \\ dx = \frac{1}{n} t^{\frac{1}{n}-1} dt \end{array} \right|$ ——

3856

$\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x \, dx$ ——

3860

$\int_0^{+\infty} x^m e^{-x^n} dx =$ ——

3863

$\int_0^{+\infty} \frac{x^{p-1} \ln x}{1+x} dx = \frac{d}{dp} \int_0^{+\infty} \frac{x^{p-1}}{1+x} dx =$ ——

3857

$\int_0^{\frac{\pi}{2}} \operatorname{tg}^n x \, dx =$ ——

Кн. р. Контр. р.

1

а) $\int_0^1 \frac{x^n dx}{\sqrt{1-x^2}}$, $(1-x^2)^{\frac{1}{2}} \underset{t_0}{\approx} 1 - \frac{1}{2}x^2 + \ldots$ т.к. дробь $\approx x^n(1+\frac{1}{2}x^2+\ldots)$

$(1+x)^{\frac{1}{2}}(1-x)^{\frac{1}{2}}$

$\Rightarrow$

сх. р.н. (пр. Вейерштр.)

б) $\int_{-\infty}^{+\infty} e^{-(x-\alpha)^2} dx$

2 Непр.

$\int_0^{\infty} \alpha e^{-\alpha x^2} dx$ пусть $\alpha \in [a, b]$ $\Rightarrow$

$$\int_0^{+\infty} \frac{e^{-at}-e^{-bt}}{t^{3/2}}\,dt,\quad \begin{matrix} a>0 \\ b>0 \end{matrix} = \left|\begin{matrix} x=t^{3/2} \\ t=x^{\frac{2}{3}} \\ dt=\frac{2}{3}x^{-\frac{1}{3}} \end{matrix}\right.$$

Классная работа

№2937

$P_n(x)=\sum_{i=0}^{n}(\alpha_i\cos ix+\beta_i\sin ix)$

Ряд Фурье: $\underset{(-l,l)}{f(x)}=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{\pi nx}{l}+b_n\sin\frac{\pi nx}{l}\right)$

$a_n=\frac{1}{l}\int_{-l}^{l}P_n(x)\cdot\cos\frac{n\pi x}{l}\,dx,\quad n=0,1,2,\dots$

$P_n(x)=\frac{2\alpha_i}{2}+\sum_{i=1}^{n}\left(\alpha_i\cos\frac{ix\pi\cdot l}{l\cdot\pi}+\beta_i\sin\frac{\pi ix\cdot l}{l\,\pi}\right)$

№2938

$f(x)=\operatorname{sgn}x\quad(-\pi<x\le\pi)$

$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\operatorname{sgn}x\cdot\cos\frac{\pi xn}{\pi}\,dx,\ n=0,1,2,\dots=\frac{1}{\pi}\left(\int_{-\pi}^{0}\cos xn\,dx+\int_{0}^{\pi}\cos xn\,dx\right)=0$

$b_n=\frac{2}{\pi}\int_0^{\pi}\operatorname{sgn}x\cdot\sin xn\,dx=\frac{2}{\pi}\left(\int_0^{\pi}\sin nx\,dx\right)=\frac{2}{n\pi}(1-(-1)^n)$

№2940

$f(x)=x$ в $(-\pi,\pi)$, $a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}x\cos xn\,dx=0$

$b_n=-\frac{2}{\pi n}\int_0^{\pi}x\,d\cos xn=-\frac{2}{\pi n}\left(\pi(-1)^n-\underbrace{\int_0^{\pi}\cos xn\,dx}_{\sin xn|_0^{\pi}}\right)=$

$=-\frac{2}{\pi n}\pi(-1)^n=\frac{2}{n}(-1)^{n+1}$

№2946

$f(x)=\sin ax\quad(-\pi,\pi)\ a\in\mathbb{R}/\mathbb{Z}$

$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\sin ax\cos xn\,dx=0$, $b_n=\frac{2}{\pi}\int_0^{\pi}\sin ax\sin nx\,dx=\frac{1}{\pi}\int_0^{\pi}(\cos(n-a)x-\cos(n+a)x)dx$

$=\frac{2\sin a\pi}{\pi}\cdot\frac{(-1)^{n-1}n}{n^2-a^2}.$

№2959

$f(x)=\sum_{n=1}^{\infty}\alpha^n\frac{\sin nx}{\sin x}$ $(|\alpha|<1)$ $a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\alpha^n\frac{\sin nx}{\sin x}\cdot\cos nx\,dx$

$\int_0^{\pi}\frac{\sin nx}{\sin x}dx=\int_0^{\pi}\frac{\sin((n-1)x+x)}{\sin x}dx=\int_0^{\pi}\left(\frac{\sin(n-1)x\cos x}{\sin x}+\frac{\sin x\cos(n-1)x}{\sin x}\right)dx$

$\int_0^{\pi}\cos(n-1)x\,dx=\frac{1}{(n-1)}\int_0^{\pi}d(\sin(n-1)x)=\frac{1}{(n-1)}(\sin(\pi-1)x+\sin x)$

через $u=\frac{\sin nx}{\sin x}$ = через exp()

a_n = через разность или сумму => о или ... => сумма Cos.

№2962

$x=2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n}$ $(-\pi<x<\pi)$, $x^2=2\int_0^x\xi\,d\xi$

$\Rightarrow x^2=4\sum(-1)^{n+1}\int_0^x\frac{\sin n\xi}{n}d\xi=4\frac{(-1)^{n+1}}{n^2}\sum_{n=1}^{\infty}(-\cos n\xi|_0^x)=4\frac{(-1)^{n+1}}{n^2}\sum_{n=1}^{\infty}(\cos nx-1)$

$x^3=3\int_0^x\xi^2\,d\xi$

1) $\frac{\pi^2x-x^3}{12}=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n^3}$. $\int\Rightarrow\sum_{n=1}^{\infty}(-1)\frac{\cos nx-1}{n^4}=\frac{2\pi^2x^2-x^4}{48}$

№2984

$f(x)$: a_n,b_n, $T=2\pi$ $\Rightarrow A_n,B_n$ $f_h(x)=\frac{1}{2h}\int_{x-h}^{x+h}f(\xi)d\xi$ – Функция Стеклова

$f(\xi)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{\pi nx}{h}+b_n\sin\frac{\pi nx}{h}\right)$ $\int_{x-h}^{x+h}f(\xi)d\xi=a_0h+\sum_{n=1}^{\infty}\frac{h a_n}{\pi n}\sin\frac{\pi n\xi}{h}-\frac{h}{\pi n}b_n\cos\frac{\pi n\xi}{h}\Big|_{x-h}^{x+h}$

замена переменных.

Д/З

№2936

$f(x)=\sin^4x$, $f(x)=(1-\cos^2x)^2=\left(1-\frac{1+\cos 2x}{2}\right)^2=\frac{3}{8}-\frac{1}{2}\cos 2x+\frac{1}{8}\cos 4x$

$a_n=\frac{1}{h}\int_{-h}^{h}\sin^4x\cdot\cos nx\,dx$, $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}\sin^4x\,dx=\frac{3}{4}$

$b_n=\frac{2}{h}\int_0^h\sin^4x\cdot\sin nx\,dx$; $a_n=\frac{2}{\pi}\int_0^{\pi}\left(\frac{3}{8}\cos nx-\frac{1}{2}\cos 2x\cos nx+\frac{1}{8}\cos 4x\cos nx\right)dx$

$2x=\frac{\alpha+\beta}{2}$, $nx=\frac{\alpha-\beta}{2}$ $\Rightarrow\frac{1}{2}(\cos(2x+nx)+\cos(2x-nx))$

№2942

$f(x)=|x|$ $x\in(-\pi,\pi)$

$a_0=\frac{2}{\pi}\int_0^{\pi} x\,dx = 2\pi$;

$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}|x|\cos nx\,dx = \frac{1}{\pi}\left(-\int_{-\pi}^{0}x\cos nx\,dx+\int_0^{\pi}x\cos nx\,dx\right) = \frac{2}{\pi}\left(\int_0^{\pi}x\cos nx+\int_0^{\pi}x\cos nx\right)$

$b_n=$ —

№2897

$f(x)=\begin{cases} ax, & \pi<x<0 \\ bx, & 0<x<\pi \end{cases}$ $a_0=\frac{2}{\pi}\int_0^{\pi} bx = $ —

$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = $ —

$b_n=$ —

№2895

$f(x)=\cos ax$, $a_0=\frac{2}{\pi}\int_0^{\pi}\cos ax = $ —

$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\underbrace{\cos ax\cos nx}_{\cos+\cos}\,dx$

$\cos + \cos$ —

№2885

$f(x)\in C$, $T=2\pi$, a_n, b_n - коэф. Фурье

$F(x)=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)f(x+t)\,dt$; $a_0=\frac{2}{\pi}\int_0^{\pi}\int_{-\pi}^{\pi}f(t)f(x+t)\,dt\,dx$

Анал-но в кн. ред.

Кл. работа

№2961 (Строим функ-ии продолж-ие - раскладываем на Фурье - на определ. части относ. с синусными

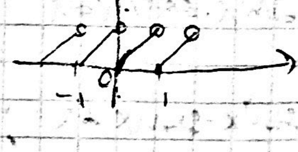
Кл.

№2955

$\frac{1}{2}-\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2\pi x n)}{n}$, $x\in\mathbb{R}$; $f(x)=x-[x]$

$x=m+r$, $r<1$

2) $f(x)=r$



$a_0=\frac{1}{2}\int_{-1}^{1}(x-[x])\,dx = 2\int_0^1 x\,dx = \frac{1}{2}$;

$a_n=\frac{1}{2}\int_{-1}^{1}(x-[x])\cos\frac{\pi x n}{1}\,dx = \int_0^1 x\cos\frac{\pi x n}{1}\,dx = \frac{2}{\pi n}\int_0^1 x\,d\sin\frac{\pi x n}{1} = \frac{2}{\pi n}\left(x\sin\frac{\pi x n}{1}\Big|_0^1\right.$

$\left.-\int_0^1 \sin\frac{\pi x n}{1}\,dx\right) = \frac{2}{\pi n}\left(0-\frac{1}{\pi n}\underbrace{\left(-\cos\frac{\pi x n}{1}\right)\Big|_0^1}\right)$

§ 2969 Аналог.

§ 2949

§ 2966

§ 2975

§ 2978

§ 2979

24 марта к/р на интегралы

~~3717(?)~~
~~3745~~
~~3716~~
~~3714(?)~~
~~3726~~
~~3728~~

~~3741~~ ~~3745~~
~~3738~~ ~~3736~~
~~3735~~ ~~3733~~

~~3753~~ 3754?
3764? +
3766?
3768

~~3776(?)~~
~~3780~~
~~3781~~
~~3785~~
~~3787~~
~~3791~~

~~3805~~
~~3807~~
3816
~~3827~~
~~3828~~
~~3811~~

Метод. оранж. стр. 27, зан. 7 „Элементы интеграла" номера д/р к/р.

~~3843~~	~~3844~~
~~3848~~	~~3845~~
3868 ?	~~3847~~
~~3870~~ ?	~~3849~~
~~3850~~	~~3856~~
~~3851~~	~~3860~~
~~3861~~	~~3865~~
~~2932~~	~~3857~~
~~2938~~	~~2936~~
~~2940~~	~~2942~~
~~2946~~	~~2973~~
~~2955~~	~~2945~~
2962 ?	2985
~~2984~~	